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PROLONGATIONS IN DIFFERENTIAL ALGEBRA

BY

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ABSTRACT

We develop the theory of higher prolongations of algebraic varieties over fields in arbitrary characteristic with commuting Hasse–Schmidt derivations. Prolongations were introduced by Buium in the context of fields of characteristic 0 with a single derivation. Inspired by work of Vojta, we give a new construction of higher prolongations in a more general context. Generalizing a result of Buium in characteristic 0, we prove that these prolongations are represented by a certain functor, which shows that they can be viewed as 'twisted jet spaces.' We give a new proof of a theorem of Moosa, Pillay and Scanlon that the prolongation functor and jet space functor commute. We also prove that the m-th prolongation and m-th jet space of a variety are differentially isomorphic by showing that their infinite prolongations are isomorphic as schemes.

Introduction

Prolongations of algebraic varieties over a differential field of characteristic 0 were introduced by Buium [Bui92], and have also been considered in more general contexts [BV95, BV96, Sca97, MPS07]. The purpose of this paper is to develop the basic theory of prolongations of algebraic varieties over fields with finitely many commuting Hasse–Schmidt (or 'higher') derivations. Let us begin by describing the idea behind Buium's construction and the connection to jet spaces of varieties. We then describe the content of the paper in more detail.

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Let (K, δ) be a differential field and let $R = K\{x_1, \ldots, x_m\}$ be the ring of differential polynomials, which is the polynomial ring in the infinitely many variables $\delta^n x_i, 0 \leq n$ and $i \leq m$. We say that $f \in R$ has order $\leq n$ if for every variable $\delta^j x_i$ that occurs in f, one has $j \leq n$. Observe that the set of elements of order $\leq n$ is a subring of R. Elements of R can be viewed as functions on affine m-space $\mathbb{A}^m = K^m$ in a natural way.

More generally, let $X \subseteq \mathbb{A}^m$ be an affine variety over (K, δ) and let A be the ring of regular functions on X. As above, there is a natural way to define the ring of differential polynomial functions of order $\leq n$ on X, called the n-th prolongation of A, which in this introduction we will denote $A^{(n)}$. Likewise, the n-th prolongation of X is $P_n(X) = \text{Spec } A^{(n)}$. Thus, there is a bijection between differential polynomial functions on X of order $\leq n$ and regular functions on $P_n(X)$. For each n, there is a natural 'projection function' $\pi_n : P_n(X) \to X$, as well as a differential polynomial map $\nabla_n : X \to P_n(X)$, which is a section of π_n. In local coordinates, given $\overline{a} \in X$, $\nabla_n(\overline{a}) = (\overline{a}, \delta(\overline{a}), \ldots, \delta^n(\overline{a}))$. Any differential polynomial function of order $\leq n$ on X factors as the composition of ∇_n with a regular function on $P_n(X)$.

One can also define the infinite prolongation $A^{(\infty)}$ of A as the ring of all differential polynomial functions on X, and the infinite prolongation $P_\infty(X)$ as Spec $A^{(\infty)}$. In this case, $A^{(\infty)}$ is naturally a differential ring, so that $X^{(\infty)}$ will be what Buium calls a $\mathcal{D}$-scheme, that is, a scheme over a differential field, equipped with a sheaf of differential rings. Note also that $P_\infty(X)$ is a pro-algebraic variety $P_\infty(X) = \varprojlim P_m(X)$. As usual, everything above globalizes to arbitrary varieties.

In an appendix to [Bui93], Buium notes that prolongations are closely related to jet (or arc) spaces of varieties, which have been studied extensively in recent years (for example, [DL99, Cra04]). Recall that the K-valued points of the n-th jet space of a variety are the $K[z]/z^{n+1}$-valued points of the variety itself. Alternatively, the n-th jet space represents a certain functor, implicit in the above characterization. What Buium observed was that his prolongations represent a twisted version of this functor. Further, given a variety that descends to the field of constants, its n-th prolongation is isomorphic to its n-th jet space. The connection between jets and prolongations does not play a significant role in Buium's theory, but it is central to the present work.

Below, we develop the theory of prolongations in a rather different way than Buium, who built on earlier work of Johnson [Joh85]. Our approach was inspired by Votja [Voj06], who gives an elegant construction of jet spaces using higher derivations. The starting point for this paper was the observation that, over a differential field, one can modify Vojta's idea so as to define prolongations in a similar manner. This perhaps further clarifies the relation between jets and prolongations. It also leads to a rather direct proof that prolongations represent the twisted jet functor introduced by Buium. We should also mention Gillet's paper [Gil02], where he develops the theory of prolongations using adjoint functors, which allows him to give new proofs of earlier results of Buium and Kolchin.

Our work was also motivated by a recent paper of Moosa, Pillay, and Scanlon [MPS07] on the model theory of differentially closed fields in characteristic 0 with finitely many commuting derivations. In that paper, the prolongation of an algebraic variety is actually defined in terms of the twisted jet functor. The authors then go on to define more generally prolongations of differential algebraic varieties, which are not treated here. We hope that our paper could be read helpfully as a companion to theirs. Good references for the model theory of differential fields include [Mar06, Pil02, Sca02]. For applications to diophantine geometry, see, for example, [HP00, PZ03, Pil04].

Let us also say something about higher derivations. These are defined below, but the basic idea is that a ring can be equipped with a sequence of additive maps, $(D_0, D_1, \ldots, D_m, \ldots)$, with D_1 a derivation and each D_m, $m > 1$, an analog of the m-th power of this derivation. In characteristic 0, they are essentially equivalent to ordinary derivations, but in positive characteristic, higher derivations are more general and rather natural to use, for example, for developing differential Galois theory [MvdP03]. From a technical point of view, also, it was more straightforward to adapt Vojta's construction to fields equipped with higher derivations.

Summary of results. In Section 1, we begin by recalling the definition of a higher derivation (of order m) from an R-algebra A to an R-algebra B. We then introduce the notion of a higher derivation over a differential ring $(R, \underline{D})$, and show that there is a universal object $\mathrm{HS}^m_{A/(R,\underline{D})}$, which is analogous to the module of Kähler differentials $\Omega_{A/R}$ in the usual case. This is the m-th prolongation of A, and we establish some basic properties of it. (In the first three sections, we restrict our attention to fields with a single higher derivation.

In Section 4, we briefly explain how to generalize our results to fields with commuting derivations.)

In Section 2, we define the m-th prolongation of a variety, which is by this point straightforward. We then prove a characterization of prolongations in terms of representable functors, a result due to Buium in characteristic 0. We also give a new proof of Moosa, Pillay, and Scanlon's theorem that the jet space functors and prolongation space functors commute.

In Section 3, we develop the foundations of Buium's theory of $\mathcal{D}$-schemes in arbitrary characteristic. Here we also prove the main result of this paper, that the m-th prolongation of a variety is isomorphic to its m-th jet space. Previously, this had only been known for $m = 1$, where the 1-st jet space is just the tangent variety.

Finally, in the last section of the paper, we develop the theory of prolongations over fields with commuting derivations, generalizing the methods in the earlier sections of the paper.

Conventions. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers, and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. Throughout the paper, the variable m will range over the ordered set $\mathbb{N} \cup \{\infty\}$, with $n < \infty$, for all $n \in \mathbb{N}$. Variables i, j, k, l will range over $\mathbb{N}$. We frequently write, for example, $i \leq m$, as shorthand for $i \leq m$, if $m \in \mathbb{N}$, and $i < m$, if $m = \infty$. Likewise, $i = 0, \ldots, m$ should be taken to mean exactly that, if $m \in \mathbb{N}$, and to mean $i = 0, 1, \ldots$, if $m = \infty$. We hope that such shorthand will not lead to any unclarity in the presentation.

All rings are commutative with unit.

1. Higher derivations

Definition 1.1 (See [Mat89] or [Voj06]): Let R be a ring, $f : R \to A$ and $R \to B$ be R-algebras, and $m \in \mathbb{N} \cup \{\infty\}$. A **higher derivation of order** m from A to B over R is a sequence $\underline{D} = (D_0, \ldots, D_m)$, or $(D_0, D_1, \ldots)$ if $m = \infty$, where $D_0 : A \to B$ is an R-algebra homomorphism and $D_1, \ldots, D_m : A \to B$ are homomorphisms of (additive) abelian groups such that

(1) $D_i(f(r)) = 0$ for all $r \in R$ and $i \geq 1$;
(2) (Leibniz Rule) for all $a, b \in A$ and $k \leq m$,

$$D_k(ab) = \sum_{i+j=k} D_i(a) D_j(b).$$

Let $\mathrm{Der}^m_R(A,B)$ denote the set of such derivations.

Higher derivations are also called **Hasse–Schmidt derivations**.

Instead of condition (1), Matsumura requires the D_i to be R-module homomorphisms, which is equivalent. Below, we will write $(D_0,\dots,D_m)$, etc. even when $m=\infty$.

Example 1.2: With R, A, B, and m as above, if $\mathbb{Q}\subseteq B\subseteq A$ and D is a usual derivation from A to B over R, then $D_i=\frac{1}{i!}D^i, i\le m$, is a higher derivation.

This is a relative notion of higher derivation. Viewing rings A and B as $\mathbb{Z}$-algebras, one also gets an absolute notion. Write $\mathrm{Der}^m(A,B)$ for $\mathrm{Der}^m_{\mathbb{Z}}(A,B)$.

Remark 1.3: Let R, A, B, and m be as above, and $\underline{D}=(D_0,\dots,D_m)$ be a sequence of maps from A to B. There is an equivalent condition for $\underline{D}$ to be a higher derivation that will be useful below.

For $m<\infty$, let $B_m=B[t]/(t^{m+1})$, the truncated polynomial ring, and for $m=\infty$, let $B_m=B[[t]]$, the ring of power series. It is easy to check that $\underline{D}$ is a higher derivation if and only if the map $g:A\to B_m$,

$$a\mapsto D_0(a)+D_1(a)t+\cdots+D_m(a)t^m$$

is a homomorphism of R-algebras. In the special case $R=\mathbb{C}$, $A=\mathbb{C}(z)$, $B=\mathbb{C}$, and $D_i=\frac{1}{i!}d^i/dz$, this says that the map taking a function $f(z)\in A$ to its (truncated) Taylor series expansion around 0 is a $\mathbb{C}$-algebra homomorphism.

Observe that $\mathrm{Der}^m_R(A,-)$ is a covariant functor from (R-algebras) to (Sets), and is represented by a (graded) R-algebra that Vojta calls $\mathrm{HS}^m_{A/R}$, which is also an A-algebra. (See also Remark 1.10.(1) below.) For $m=1$, $\mathrm{HS}^m_{A/R}$ is just the symmetric algebra on $\Omega_{A/R}$.

Definition 1.4: Let A be an R-algebra. A **higher derivation** on A is a sequence of maps $\underline{D}\in\mathrm{Der}^\infty_R(A,A)$ such that $D_0=\mathrm{Id}_A$. In this case, we call $(A,\underline{D})$ a **$\mathcal{D}$-ring over** R. A **homomorphism** $f:(A,\underline{D})\to(B,\underline{D})$ between $\mathcal{D}$-rings over R is an R-algebra homomorphism such that $f(D_i(a))=D_i(f(a))$, for all $a\in A$ and all i. We will often be interested in the case when A is just a ring (that is, a $\mathbb{Z}$-algebra) and call $(A,\underline{D})$ simply a $\mathcal{D}$-ring. The set of **constants** of $(A,\underline{D})$ are those $a\in A$ such that $D_ia=0$, for all $i\ge1$. Given $(A,\underline{D})$, say that $\underline{D}$ is **iterative** if for all i,j, $D_i\circ D_j=\binom{i+j}{j}D_{i+j}$.

Below we will only consider iterative $\mathcal{D}$-rings. Note that it does not make sense to talk of a higher derivation from one ring to another being iterative.

Remark 1.5: As above, let $A_m = A[t]/(t^{m+1})$, for $m < \infty$, and $A_\infty = A[[t]]$. Let $h : A_m \to A$ be the homomorphism sending $f(t) \in A_m$ to $f(0) \in A$. Then a sequence of maps $\underline{D} = (D_0, \dots, D_m)$ from A to A is a higher derivation if and only if the map $d : A \to A_m$, with $d(a) = \sum_{i \le m} D_i(a)t^i$, is a homomorphism and $h \circ d = \mathrm{Id}_A$.

Derivations on a ring extend uniquely to localizations (see [Mat89], Theorem 27.2, or [Oku87], Section 1.6, Theorem 1).

LEMMA 1.6 (Quotient Rule): *Let $(A, \underline{D})$ be a $\mathcal{D}$-ring. For all invertible $b \in A$, and $n \in \mathbb{N}^+$,*

$$D_n\left(\frac{1}{b}\right) = \frac{-1}{b}\left(\sum_{i<n} D_i(b) \cdot D_{n-i}\left(\frac{1}{b}\right)\right).$$

To obtain this, observe $0 = D_n(1) = D_n(b \cdot \frac{1}{b}) = \sum_{i \le n} D_i(b) \cdot D_{n-i}(\frac{1}{b}))$, and solve for $D_n(\frac{1}{b})$.

LEMMA 1.7: *Let $(R, \underline{D})$ be a $\mathcal{D}$-ring, and S a multiplicative subset of R. Then there is a unique extension of $\underline{D}$ to $S^{-1}R$.*

We now introduce higher derivations on R-algebras when $(R, \underline{D})$ is also a $\mathcal{D}$-ring. This is closely related to Buium's prolongations, where (R, δ) is a differential ring, A is an R-algebra, and one considers derivations on A that are 'compatible' with δ.

Definition 1.8: Let $(R, \underline{D})$ be a $\mathcal{D}$-ring. An R-algebra A, given by $f : R \to A$, is an **$(R, \underline{D})$-algebra** if for all $r \in R$ and all i, $f(r) = 0$ implies $f(D_i(r))$=0; in other words, $\mathrm{Ker}(f)$ is a $\mathcal{D}$-ideal.

Let $f : R \to A$ and B be $(R, \underline{D})$-algebras. A **higher derivation** from A to B of order m over $(R, \underline{D})$ is a sequence $\underline{\delta} = (\delta_0, \dots, \delta_m)$ such that $\delta_0 : A \to B$ is an R-algebra homomorphism, $\delta_i : A \to B, 1 \le i \le m$, are (additive) abelian group homomorphisms, and

(1) $\delta_i(f(r)) = \delta_0(f(D_i(r)))$, for $r \in R$;
(2) (Leibniz Rule) $\delta_k(ab) = \sum_{i+j=k} \delta_i(a)\delta_j(b)$, for $a, b \in A$.

Let $\mathrm{Der}^m_{(R,\underline{D})}(A, B)$ denote the set of such derivations.

Note that if $(R, \underline{D})$ is trivial, that is, $D_0 = \mathrm{Id}_R$ and $D_i = 0$, $i \geq 1$, then this reduces to Definition 1.1.

As above, given an $(R, \underline{D})$-algebra A, $\mathrm{Der}^m_{(R,\underline{D})}(A, -)$ is a covariant functor from ($(R, \underline{D})$-algebras) to (Sets), which we will now observe to be representable.

Definition 1.9: Let $(R, \underline{D})$ be a $\mathcal{D}$-ring, $f : R \to A$ an $(R, \underline{D})$-algebra. For all m, define $\mathrm{HS}^m_{A/(R,\underline{D})}$ to be the A-algebra that is the quotient of the polynomial algebra $A[x^{(i)}]_{x \in A, 1 \leq i \leq m}$ by the ideal I generated by:

(1) $(x + y)^{(i)} - x^{(i)} - y^{(i)} : x, y \in A, i = 1, \ldots, m$;
(2) $(xy)^{(k)} - \sum_{i+j=k} x^{(i)} y^{(j)} : x, y \in A, k = 1, \ldots, m$;
(3) $f(r)^{(i)} - f(D_i(r)) : r \in R, i = 1, \ldots, m$.

In $A[x^{(i)}]$, we identify $x \in A$ with $x^{(0)}$. There is a universal derivation $\underline{d} = (d_0, \ldots, d_m) : A \to \mathrm{HS}^m_{A/(R,\underline{D})}$ such that for $i \leq m$ and $x \in A, d_i(x) = x^{(i)}$.

Remarks 1.10: (1) With the above notation, if $(R, \underline{D})$ is a trivial $\mathcal{D}$-ring, then $\mathrm{HS}^m_{A/(R,\underline{D})}$ is the same as Vojta's $\mathrm{HS}^m_{A/R}$. In general, though, $\mathrm{HS}^m_{A/(R,\underline{D})}$ is not naturally graded, because of condition (3).
(2) For $m = 1$, we get the first prolongation in the sense of Buium.
(3) For $0 \leq m < n \leq \infty$, there are natural A-algebra homomorphisms $f_{mn} : \mathrm{HS}^m_{A/(R,\underline{D})} \to \mathrm{HS}^n_{A/(R,\underline{D})}$. These form a directed system, and

$$\mathrm{HS}^\infty_{A/(R,\underline{D})} = \varinjlim_{i \in \mathbb{N}} \mathrm{HS}^i_{A/(R,\underline{D})}.$$

Definition 1.11: Let $(R, \underline{D})$ be a $\mathcal{D}$-ring. A $\mathcal{D}$-$(R, \underline{D})$-**algebra** is a $\mathcal{D}$-ring $(A, \underline{D})$ that is also an $(R, \underline{D})$-algebra via some map $f : R \to A$, such that the derivation on A is compatible with that on R. That is, for all $r \in R$ and all i, $D_i(f(r)) = f(D_i(r))$.

Lemma 1.12: *Given an $(R, \underline{D})$-algebra A, there is a canonical way to make $\mathrm{HS}^\infty_{A/(R,\underline{D})}$ into a $\mathcal{D}$-$(R, \underline{D})$-algebra.*

Proof. Extend the universal derivation $\underline{d} : A \to \mathrm{HS}^\infty_{A/(R,\underline{D})}$ to an (iterative) higher derivation on $\mathrm{HS}^\infty_{A/(R,\underline{D})}$ by setting

$$d_i\left(x^{(j)}\right) = \binom{i+j}{i} x^{(i+j)}. \qquad \blacksquare$$

Definition 1.13: Let $(R, \underline{D})$ be a $\mathcal{D}$-ring. Let

$$R_m = \begin{cases} R[t]/(t^{m+1}) & \text{for } m < \infty \\ R[[t]] & \text{for } m = \infty \end{cases}$$

For each m, we define a 'twisted' homomorphism $e : R \to R_m$ by $e(r) = D_0(r) + D_1(r)t + \cdots + D_m(r)t^m$. Let $\tilde{R}_m$ be the R-algebra isomorphic to R_m as a ring, and made into an R-algebra via the map $e : R \to R_m$.

Let $f : R \to B$ be an $(R, \underline{D})$-algebra. Define $B_m = B[t]/(t^{m+1})$, for $m < \infty$, and $B_\infty = B[[t]]$. Let $\tilde{B}_m$ be the ring B_m made into an R-algebra via the map $\tilde{f} : R \to \tilde{B}_m$ that sends

$$r \mapsto f(D_0(r)) + f(D_1(r))t + \cdots + f(D_m(r))t^m.$$

PROPOSITION 1.14: *Let $(R, \underline{D})$ be a $\mathcal{D}$-ring. For all m, R_m and $\tilde{R}_m$ are isomorphic as R-algebras.*

Proof. Suppose first that $m < \infty$. We claim that the map $\psi : R_m \to \tilde{R}_m$ with $\psi(r) = e(r) = D_0(r) + D_1(r)t + \cdots + D_m(r)t^m$, for $r \in R$, and $\psi(t) = t$ is an isomorphism of R-algebras. Clearly, ψ is a homomorphism, so it suffices to check that it is injective and surjective.

Let $a = a_0 + a_1 t + \cdots + a_m t^m$, so $\psi(a) = e(a_0) + e(a_1)t + \cdots + e(a_m)t^m$. Rearranging terms, one gets

$$\psi(a) = a_0 + (D_1(a_0) + a_1)t + (D_2(a_0) + D_1(a_1) + a_2)t^2 + \cdots + (D_m(a_0) + \ldots + a_m)t^m.$$

Suppose that $\psi(a) = 0$, so in particular each coefficient of $\psi(a)$ as a polynomial in t is 0. Thus, $a_0 = 0$. Looking at the next term, $0 = D_1(a_0) + a_1 = a_1$. Continuing this way, one sees that all of the a_i's are 0, so a itself is 0 and ψ is injective.

To show that ψ is surjective, it suffices to show that for each $r \in R$, $r = r + 0t + \cdots + 0t^m \in \tilde{R}_m$ is in $\mathrm{Im}(\psi)$. (Of course, $r \neq \psi(r)$.) For fixed r, we iteratively define a sequence, $c_0, c_1, \ldots, c_m$, of elements of R_m with the following properties. One, for all $i \leq m$, the constant term of $\psi(c_i)$, as a polynomial in t, is r. Two, for $i \geq 1$, and $1 \leq j \leq i$, the coefficient of t^j in $\psi(c_i)$ is 0. Then $\psi(c_m) = r$, as desired. Set $c_0 = r$. For the iterative step, suppose that $c_0, \ldots, c_i$ have been defined, and that $\psi(c_i) = r + a_{i+1}t^{i+1} + \cdots + a_m t^m$. Let $c_{i+1} = c_i - a_{i+1}t^{i+1}$. Clearly, this procedure yields such a sequence.

For $m = \infty$, given the isomorphisms $\psi_i : R_i \to \tilde{R}_i$, $i < \infty$, it suffices to note that R_∞ and $\tilde{R}_\infty$ are the inverse limits of $\{R_i\}_{i<\infty}$ and $\{\tilde{R}_i\}_{i<\infty}$, respectively. The required isomorphism $\psi_\infty : R_\infty \to \tilde{R}_\infty$ is again given by sending $r \in R$ to $e(r)$, and sending t to t. ∎

More generally, we have the following.

Proposition 1.15: *Let $(R, \underline{D})$ be a $\mathcal{D}$-ring, and B an $(R, \underline{D})$-algebra such that $\underline{D}$ extends to a derivation on B. Then $\tilde{B}_m \cong B_m$, as R-algebras.*

Proof. Choose a derivation $\underline{D}$ on B extending $(R, \underline{D})$, and then argue as above. ∎

For fields of characteristic 0, any (higher) derivation on a field K can be extended to a derivation on any extension field $L \supseteq K$, so one has the following corollary.

Corollary 1.16: *Let $(K, \underline{D})$ be a $\mathcal{D}$-field of characteristic 0 and $L \supseteq K$ an extension field. Then $\tilde{L}_m \cong L_m$, as K-algebras.*

On the other hand, L_m and $\tilde{L}_m$ are not always isomorphic.

Proposition 1.17: *Let $(K, \underline{D})$ be a $\mathcal{D}$-field of characteristic $p > 0$, and let L be a purely inseparable algebraic extension of K, such that there is an $a \in L, b \in K$, with $a^p = b$ and $D_1(b) \neq 0$. Then for each $m \geq 1$, L_m and $\tilde{L}_m$ are not isomorphic as K-algebras.*

Proof. We show that there is no K-algebra homomorphism from L to $\tilde{L}_m$, which immediately implies the proposition. In particular, we argue that any such homomorphism would give an extension of D_1 to a derivation on L, which is impossible (as $0 \neq D_1(b) = D_1(a^p) = pa^{p-1}D_1(a) = 0$, contradiction). Suppose that $\phi : L \to \tilde{L}_m$ is a K-algebra homomorphism. For all $c \in K$, $\phi(c) = D_0(c) + D_1(c)t + \cdots + D_m(c)t^m$. For $x \in L$, write $\phi(x) = \phi_0(x)+\phi_1(x)t+\cdots+\phi_m(x)t^m$, with $\phi_i : L \to L$, for $i = 0, \ldots, m$. We claim that for all $x \in L$, $\phi_0(x) = x$. Indeed, this is clear for $x \in K$, as $\phi_0(x) = D_0(x) = x$. Otherwise, $x^{p^n} = y$, for some n and some $y \in K$. Then

$$\phi(x)^{p^n} = (\phi_0(x) + \phi_1(x)t + \cdots + \phi_m(x)t^m)^{p^n} = \phi_0(x)^{p^n} + t \cdot g(t)$$

and also

$$\phi(x)^{p^n} = \phi\big(x^{p^n}\big) = \phi(y) = D_0(y) + t \cdot h(t)$$

with $g(t), h(t)$ polynomials in $L[t]$. Thus $\phi_0(x)^{p^n} = D_0(y) = y$, so $\phi_0(x) = x$, as desired.

By the claim, $(L, \underline{\phi})$ is a higher derivation of order m that extends $(K, \underline{D})$. In particular, ϕ_1 is an extension of D_1 to L, which is impossible. ∎

PROPOSITION 1.18: *Let $(R, \underline{D})$ be a $\mathcal{D}$-ring, $R \to A$ and $R \to B$ be $(R, \underline{D})$-algebras. Given a higher derivation $\underline{\delta} = (\delta_0, \ldots, \delta_m) : A \to B$ there exists a unique $(R, \underline{D})$-algebra homomorphism, $\phi : \mathrm{HS}^m_{A/(R,\underline{D})} \to B$ such that $(\delta_0, \ldots, \delta_m) = (\phi \circ d_0, \ldots, \phi \circ d_m)$. Thus $\mathrm{HS}^m_{A/(R,\underline{D})}$ (together with the universal derivation $\underline{d} : A \to \mathrm{HS}^m_{A/(R,\underline{D})}$) represents the functor $\mathrm{Der}^m_{(R,\underline{D})}(A, -)$.*

Proof. Define $\phi_0 : A[x^{(i)}]_{x \in A, i=1,\ldots,m} \to B$ by $x^{(i)} \mapsto \delta_i(x)$. By the construction of the ideal $I \subseteq A[x^{(i)}]$ and properties of derivation, we get that $\mathrm{Ker}(\phi_0) \supseteq I$, so there is an induced map $\phi : \mathrm{HS}^m_{A/(R,\underline{D})} \to R$. As $\underline{\delta} = \phi \circ \underline{d}$, ϕ is unique. Thus the map

$$\mathrm{Hom}_R(\mathrm{HS}^m_{A/(R,\underline{D})}, B) \longrightarrow \mathrm{Der}^m_{(R,\underline{D})}(A, B)$$

is bijective. ∎

Compare the following proposition to Remark 1.3.

PROPOSITION 1.19: *Let $(R, \underline{D})$ be a $\mathcal{D}$-ring, $f : R \to A, g : R \to B$ be $(R, \underline{D})$-algebras. Given a derivation $\underline{\delta} = (\delta_0, \ldots, \delta_m) \in \mathrm{Der}^m_{(R,\underline{D})}(A, B)$, define a map $\phi = \phi_{\underline{\delta}} : A \to \tilde{B}_m$ by $\phi(a) = \delta_0(a) + \delta_1(a)t + \cdots + \delta_m(x)t^m$. Then $\phi_{\underline{\delta}} \in \mathrm{Hom}_R(A, \tilde{B}_m)$ and the map*

$$\underline{\delta} \mapsto \phi_{\underline{\delta}} : \mathrm{Der}^m_{(R,\underline{D})}(A, B) \to \mathrm{Hom}_R(A, \tilde{B}_m)$$

is a bijection.

Proof. The δ_i are homomorphism of the additive groups, so ϕ is also. The Leibniz Rule implies that ϕ is multiplicative, so it only remains to show that $\phi \circ f = \tilde{g}$, where $\tilde{g} : R \to \tilde{B}_m$ is the homomorphism that makes $\tilde{B}_m$ into an R-algebra. Check,

$$\begin{aligned}\phi \circ f(x) &= \delta_0(f(x)) + \delta_1(f(x))t + \cdots + \delta_m(f(x))t^m \\ &= \delta_0(f(x)) + \delta_0(f(D_1(x)))t + \ldots + \delta_0(f(D_m(x)))t^m \\ &= g(x) + g(D_1(x))t + \ldots + g(D_m(x))t^m = \tilde{g}(x)\end{aligned}$$

This establishes injectivity. To show surjectivity, we just reverse the direction of the argument. Suppose that $h : A \to \tilde{B}_m$ is an R-algebra homomorphism, which we can write as $h(a) = h_0(a) + h_1(a)t + \cdots + h_m(a)t^m$, each h_i a map from A to B. We claim that $\{h_i : i \leq m\}$ is a higher derivation from A to B. Clearly the h_i are additive and satisfy the Leibniz Rule. So it suffices to show that for $r \in R$ and $i \leq m$, $h_i(f(r)) = h_0(f(D_i(r)))$. Since h is an R-algebra homomorphism, one has $h_i(f(r)) = h_0(f(D_i(r)) = g(D_i(r))$. ∎

The next corollary follows immediately from Proposition 1.18 and Proposition 1.19. It is the main point in the characterization of prolongations in terms of representable functors.

Corollary 1.20 (Buium): *There is a natural bijection*

$$\mathrm{Hom}_R(\mathrm{HS}^m_{A/(R,\underline{D})}, B) \longrightarrow \mathrm{Hom}_R(A, \tilde{B}_m)$$

The next result is due to Buium [Bui93] and Gillet [Gil02] in a slightly different context. In fact, Gillet defines the prolongation functor to be the left adjoint of the forgetful functor from differential algebras to algebras.

Proposition 1.21: *Let $(R, \underline{D})$ be a $\mathcal{D}$-ring, Alg_R be the category of $(R, \underline{D})$-algebras, and $\mathcal{D}$-Alg_R be the category of $\mathcal{D}$-$(R, \underline{D})$-algebras. Let U be the forgetful functor $\mathcal{D}$-$\mathrm{Alg}_R \to \mathrm{Alg}_R$. Then the functor $F : \mathrm{Alg}_R \to \mathcal{D}$-$\mathrm{Alg}_R$, sending A to $\mathrm{HS}^\infty_{A/(R,\underline{D})}$, is the left adjoint of U.*

Proof. Essentially immediate from the explicit construction given of $\mathrm{HS}^\infty_{A/(R,\underline{D})}$. That is, given an R-algebra map $f : A \to (B, \underline{D}^B)$, there is an obvious, unique way to lift f to a $\mathcal{D}$-$(R, \underline{D})$-algebra map $f^\infty : \mathrm{HS}^\infty_{A/(R,\underline{D})} \to (B, \underline{D}^B)$. For example, for $x^{(i)} \in \mathrm{HS}^\infty_{A/(R,\underline{D})}$, $x \in A$, then $f^\infty(x^{(i)}) = D^B_i(f(x))$. ∎

The next result is what Vojta calls the second fundamental exact sequence, adapted to our context. For completeness, we include his proof, which carries over directly.

Proposition 1.22 (Second fundamental exact sequence): *Let $(R, \underline{D})$ be a $\mathcal{D}$-ring and $R \to A \to B$ a sequence of ring homomorphisms. Assume that $A \to B$ is surjective, and let I be its kernel. Let J be the ideal in $\mathrm{HS}^m_{A/(R,\underline{D})}$ generated by $\{d_i x : i \leq m, x \in I\}$. Then the following sequence is exact.*

$$0 \longrightarrow J \longrightarrow \mathrm{HS}^m_{A/(R,\underline{D})} \longrightarrow \mathrm{HS}^m_{B/(R,\underline{D})} \longrightarrow 0$$

In the definition of J, it suffices to let x vary over a set of generators of I.

Proof. Exactness on the left is immediate. The natural map $h : \mathrm{HS}^m_{A/(R,\underline{D})} \longrightarrow \mathrm{HS}^m_{B/(R,\underline{D})}$ is surjective and its kernel contains J, so it remains to show that $\mathrm{Ker}(h) = J$.

From the definition of HS^m, we have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & A[x^{(i)}]_{x\in A, i=1,\ldots,m} & \longrightarrow & \mathrm{HS}^m_{A/(R,\underline{D})} & \longrightarrow & 0 \\ & & \big\downarrow f & & \big\downarrow g & & \big\downarrow h & & \\ 0 & \longrightarrow & K' & \longrightarrow & B[x^{(i)}]_{x\in B, i=1,\ldots,m} & \longrightarrow & \mathrm{HS}^m_{B/(R,\underline{D})} & \longrightarrow & 0 \end{array}$$

By Definition 1.9, the map f is surjective, so by the Snake Lemma, $\mathrm{Ker}(g)$ maps onto $\mathrm{Ker}(h)$. But $\mathrm{Ker}(g)$ is generated by

$$I \cup \{d_i x - d_i y : i = 1, \ldots, m;\ x, y \in A;\ x - y \in I\}.$$

This implies that the kernel of h is generated by the set

$$\{d_i x : i = 0, \ldots, m,\ x \in I\},$$

as desired. ∎

The next two results also occur in Vojta [Voj06].

Proposition 1.23: *Let $(R, \underline{D})$ be a $\mathcal{D}$-ring, and $A = R[x_i]_{i\in I}$. Then $\mathrm{HS}^m_{A/(R,\underline{D})}$ is the polynomial algebra $A[d_j x_i]_{i\in I, j=1,\ldots,m}$.*

Proof. Essentially obvious, but also proved in [Voj06]. ∎

Corollary 1.24: *Let A be an $(R, \underline{D})$-algebra, $A \cong R[x_i]_{i\in I}/(f_j)_{j\in J}$. Then*

$$\mathrm{HS}^m_{A/(R,\underline{D})} \cong A[d_k x_i]_{i\in I, k=1,\ldots,m}/(d_k f_j)_{j\in J, k=1,\ldots,m}.$$

Suppose further that all of the coefficients of the polynomials $f_j, j \in J$, are constants in the ring R. Then $\mathrm{HS}^m_{A/(R,\underline{D})}$ is the same as $\mathrm{HS}^m_{A/R}$, as defined by Vojta.

Proof. The first statement follows from Propositions 1.22 and 1.23. The second follows from the first, and the analogous statement from [Voj06]. ∎

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2. Prolongations

In this section, we assume throughout that $(K, \underline{D})$ is a $\mathcal{D}$-field. Probably everything also works over $\mathcal{D}$-rings. We define prolongations of schemes/varieties over $(K, \underline{D})$. In characteristic 0, we essentially get Buium's prolongations, though there is a slight difference since we are using higher derivations. The construction of the prolongations is direct, but the results of the previous section provide the connection with representable functors. In characteristic 0, this agrees with Buium, but in characteristic $p > 0$, it avoids problems that would arise if one tries to adapt Buium directly to characteristic p, involving 'dividing by p.'

Lemma 2.1: *Let A be a $(K, \underline{D})$-algebra and S a multiplicative subset of A. Then there is an isomorphism*

$$\mathrm{HS}^m_{A/(K,\underline{D})} \otimes_A S^{-1}A \longrightarrow \mathrm{HS}^m_{S^{-1}A/(K,\underline{D})}.$$

Proof. For all $a \in A$, let $\overline{a}$ denote the image of a in $S^{-1}A$ under the canonical map. We show that the natural map ϕ that sends $d_i a \otimes s^{-1}b$ to $s^{-1}b \cdot d_i\overline{a}$, for all $a, b \in A, s \in S$, and $i \leq m$, is an isomorphism. Clearly, ϕ is a homomorphism. By the quotient rule, for $i \leq m$ and $s^{-1}\overline{b} \in S^{-1}A$, $d_i(s^{-1}\overline{b})$ can be written as $s^{-n}c$, for some $c \in \mathrm{HS}^m_{A/(K,\underline{D})}$, so ϕ is surjective. To show that ϕ is injective, it suffices to define its inverse. Let $s^{-1}\overline{b} \in S^{-1}A$. We want to define $\phi^{-1}(d_i(s^{-1}\overline{b}))$ as $c \otimes s^{-n}$, but for this we need to check that if $s^{-1}\overline{b} = t^{-1}\overline{c}$ in $S^{-1}A$, then $\phi^{-1}(d_i(s^{-1}\overline{b})) = \phi^{-1}(d_i(t^{-1}\overline{c}))$, for all $i \leq m$. To simplify the presentation, let us assume that $s = t = 1$. As $\overline{b} = \overline{c}$ in $S^{-1}A$ is equivalent to there being an $s \in S$ such that $s(c-b) = 0$ in A, it suffices for us to show that for all $a \in A$, if $\overline{a} = 0$ in $S^{-1}A$, that is, there is $s \in S$ such that $sa = 0$ in A, then $d_i a \otimes 1 = 0$ in $\mathrm{HS}^m_{A/(K,\underline{D})} \otimes_A S^{-1}A$, for $i \leq m$.

We argue by induction on i. The case $i = 0$ is obvious, so assume that we have proved that $d_j a \otimes 1 = 0$ in $\mathrm{HS}^m_{A/(K,\underline{D})} \otimes_A S^{-1}A$, for all $j < i$. Then

$$0 = sa \otimes 1 = d_i(sa) \otimes 1 = \sum_{j+k=i} (d_j s d_k a \otimes 1) = s d_i a \otimes 1 = (d_i a \otimes 1)(1 \otimes s),$$

so $(d_i a \otimes 1) = 0$ in $\mathrm{HS}^m_{A/(K,\underline{D})} \otimes_A S^{-1}A$, as desired. ∎

The next theorem is the differential version of Theorem 4.3 of [Voj06], and is an easy consequence of Lemma 2.1, exactly as in Vojta.

THEOREM 2.2: *Let X be a K-scheme. For all m, there exists a sheaf of $\mathcal{O}_X$-algebras $\mathrm{HS}^m_{X/(K,\underline{D})}$ such that (i) for each open affine* $\operatorname{Spec} A \subseteq X$*, there is an isomorphism*

$$\phi_A : \Gamma(\operatorname{Spec} A, \mathrm{HS}^m_{X/(K,\underline{D})}) \longrightarrow \mathrm{HS}^m_{A/(K,\underline{D})}$$

of $(K, \underline{D})$-algebras, and (ii) the various ϕ_A are compatible with the localization isomorphism of Lemma 2.1. Moreover, the collection $((\mathrm{HS}^m_{X/(K,\underline{D})}), (\phi_A)_A)$ is unique.

Definition 2.3: Let X be a K-scheme. For all m, the m-th **prolongation** of X is the scheme

$$P_m(X/(K,\underline{D})) := \mathbf{Spec}\, \mathrm{HS}^m_{X/(K,\underline{D})}.$$

Suppose that A is a $(K, \underline{D})$-algebra. We write

$$P_m(A/(K,\underline{D})) = P_m(\operatorname{Spec} A/(K,\underline{D})),$$

which equals $\mathbf{Spec}\, \mathrm{HS}^m_{A/(K,\underline{D})}$.

We will also write $X^{(m)}$ or $P_m(X)$ for $P_m(X/(K,\underline{D}))$.

(For the definition of **Spec**, see, for example, [Har77] Ch. II, Ex. 5.17.)

Recall that $K_m = K[t]/(t^{m+1})$, for $m < \infty$, $K_\infty = K[[t]]$, and that $e : K \to K_m$ denotes the twisted homomorphism. We also let $e : \operatorname{Spec} K_m \to \operatorname{Spec} K$ denote the corresponding twisted morphism of schemes. Given a K-scheme Y, let $(Y \times_K \operatorname{Spec} K_m)\tilde{}$ denote the scheme $(Y \times_K \operatorname{Spec} K_m)$ made into a K-scheme via the map $e \circ p : (Y \times_K \operatorname{Spec} K_m) \to \operatorname{Spec} K$, where $p : (Y \times_K \operatorname{Spec} K_m) \to \operatorname{Spec} K_m$ is the canonical projection.

THEOREM 2.4 (Buium): *Let X be a K-scheme. For all m, the scheme $P_m(X)$ represents the functor from K-schemes to sets given by*

$$Y \mapsto \operatorname{Hom}_K((Y \times_K \operatorname{Spec} K_m)\tilde{}, X).$$

Proof. For X and Y affine, this follows immediately from Corollary 1.20. The general case follows by gluing affines. ∎

Recall that given a K-scheme X, the m-th jet space of X, which we denote $J_m(X)$, is the scheme that represents the following functor from K-schemes to sets.

$$Y \mapsto \mathrm{Hom}_K(Y \times_K \operatorname{Spec} K_m, X)$$

Buium's theorem clarifies the relationship between prolongations and jets. One also has the following fact, due again to Buium.

PROPOSITION 2.5: *Let X be a $(K, \underline{D})$-scheme such that $X = X' \times_C \operatorname{Spec} K$, where C is the field of constants of K, and X' is some C-scheme. (That is, X descends to, or is defined over, C.) Then for all m, $P_m(X) \cong J_m(X)$.*

Proof. This follows from Corollaries 1.24 and 2.10, below, and the description of jets in [Voj06] (see, for example, Theorem 4.3 and Definition 4.4). ∎

The next result, due to Moosa, Pillay, and Scanlon, generalizes the well-known fact that for all $m, n \leq \infty$, $J_m(J_n(X)) = J_n(J_m(X))$, which can be seen by observing that they represent the same functor. In the original version of [MPS07] it was stated without proof. A revised version contains a proof using the Weil restriction.

THEOREM 2.6 (Moosa, Pillay, and Scanlon): *Let X be a K-scheme. For all $m, n \leq \infty$,*

$$J_m(P_n(X)) \cong P_n(J_m(X)).$$

Proof. We include two proofs. The first is direct and uses the construction of jets and prolongations from [Voj06] and this paper. The second, closer in spirit to [MPS07], shows that $J_m(P_n(X))$ and $P_n(J_m(X))$ represent the same functor.

It suffices to prove this for affine schemes, so assume that $X = \operatorname{Spec} A$. Even though K is a differential field, we will use $\mathrm{HS}^m_{A/K}$ to denote the A-algebra defined by Vojta, which is defined exactly like $\mathrm{HS}^m_{A/(K,\underline{D})}$ in Definition 1.9, except that one replaces condition (3) with

$$f(r)^{(i)} : r \in K, \quad i = 1, \ldots, m.$$

The point from our perspective is that $\operatorname{Spec} \mathrm{HS}^m_{A/K}$ is the m-th jet space of X, while $\operatorname{Spec} \mathrm{HS}^m_{A/(K,\underline{D})}$ is the m-th prolongation of X. Thus

$$J_m(P_n(X)) = \operatorname{Spec} \mathrm{HS}^m_{\mathrm{HS}^n_{A/(K,\underline{D})}/K}$$

and

$$P_n(J_m(X)) = \operatorname{Spec} \mathrm{HS}^n_{\mathrm{HS}^m_{A/K}/(K,\underline{D})}.$$

We want to show that the two K-algebras above are isomorphic. Let us use $\underline{d}$ for the universal derivation on HS^m, corresponding to jets, and $\underline{\delta}$ for the universal derivation on HS^n, corresponding to prolongations. An arbitrary element of $\mathrm{HS}^m_{\mathrm{HS}^n_{A/(K,\underline{D})}/K}$ can be written as a sum of terms

$$d_i\delta_j a: \quad i \leq m, \quad j \leq n, \quad a \in A,$$

and an arbitrary element of $\mathrm{HS}^n_{\mathrm{HS}^m_{A/K}/(K,\underline{D})}$ as a sum of terms

$$\delta_j d_i a: \quad i \leq m, \quad j \leq n, \quad a \in A.$$

We claim that the K-algebra morphism $\theta : \mathrm{HS}^m_{\mathrm{HS}^n_{A/(K,\underline{D})}/K} \to \mathrm{HS}^n_{\mathrm{HS}^m_{A/K}/(K,\underline{D})}$ with $\theta(d_i\delta_j a) = \delta_j d_i a$ is an isomorphism.

First we check that θ is well-defined. For example,

$$d_i\left(\delta_j(a+b) - \delta_j(a) - \delta_j(b)\right) = 0,$$

so we check

$$\theta\left(d_i(\delta_j(a+b) - \delta_j(a) - \delta_j(b))\right) = \delta_j(d_i(a+b) - d_i(a) - d_i(b)) = 0.$$

Likewise,

$$\begin{aligned} 0 &= d_i(\delta_j(ab) - \sum_{k+l=j} \delta_k(a)\delta_l(b) = d_i\delta_j(ab) - \sum_{k+l=j} d_i(\delta_k(a)\delta_l(b)) \\ &= d_i\delta_j(ab) - \sum_{k+l=j}\sum_{m+n=i} d_m\delta_k(a)d_n\delta_l(b). \end{aligned}$$

And we check

$$\begin{aligned} \theta\Bigg(d_i\delta_j(ab) - \sum_{k+l=j}\sum_{m+n=i} d_m\delta_k(a)d_n\delta_l(b)\Bigg) &= \delta_j d_i(ab) - \sum_{m+n=i}\sum_{k+l=j} \delta_k d_m(a)\delta_l d_n(b) \\ &= \delta_j d_i(ab) - \sum_{m+n=i} \delta_j(d_m(a)d_n(b)) \\ &= \delta_j\Bigg(d_i(ab) - \sum_{m+n=i} d_m(a)d_n(b)\Bigg) = 0. \end{aligned}$$

Finally, for $c \in K$, we have $0 = d_i(\delta_j(c) - \delta_0 D_j(c))$. For $i \geq 1$,

$$\theta(d_i(\delta_j(c) - \delta_0 D_j(c))) = \delta_j d_i(c) - \delta_0 d_i(D_j(c)) = 0$$

and, for $i = 0$,

$$\theta(d_0(\delta_j(c) - \delta_0 D_j(c))) = \delta_j d_0(c) - \delta_0 d_0(D_j(c)) = \delta_j(c) - \delta_0(D_j(c)) = 0.$$

Now that we know that θ is well-defined, it is clear from the definition that it respects sums and products. Finally, it is clearly a bijection, since there is an obvious inverse.

We now give a second proof, along the lines of [MPS07]. Let Y be a K-scheme. There are natural bijections

$$\begin{aligned}\operatorname{Hom}_K(Y, J_m(P_n(X))) &\simeq \operatorname{Hom}_K(Y \times_K \operatorname{Spec} K_m, P_n(X)) \\ &\simeq \operatorname{Hom}_K(((Y \times_K \operatorname{Spec} K_m) \times_K \operatorname{Spec} K_n)\tilde{}, X)\end{aligned}$$

and also natural bijections

$$\begin{aligned}\operatorname{Hom}_K(Y, P_n(J_m(X))) &\simeq \operatorname{Hom}_K((Y \times_K \operatorname{Spec} K_n)\tilde{}, J_m(X)) \\ &\simeq \operatorname{Hom}_K(((Y \times_K \operatorname{Spec} K_n)\tilde{} \times_K \operatorname{Spec} K_m), X)\end{aligned}$$

Thus, it suffices to show that for all Y,

$$((Y \times_K \operatorname{Spec} K_m) \times_K \operatorname{Spec} K_n)\tilde{} \cong ((Y \times_K \operatorname{Spec} K_n)\tilde{} \times_K \operatorname{Spec} K_m).$$

In fact, it suffices to prove this for Y affine. We rephrase this as a question about isomorphisms of K-algebras. Given a K-algebra C, let us write $(C \otimes_K K_n)\tilde{}$ for what we called $\tilde{C}_n$ in Definition 1.13. This more closely parallels our notation for schemes. That is, for $Z = \operatorname{Spec} C$, then $(Z \times_K \operatorname{Spec} K_n)\tilde{} = \operatorname{Spec}((C \otimes_K K_n)\tilde{})$.

Everything reduces to showing that, for all $Y = \operatorname{Spec} B$, the following are isomorphic.

$$((B \otimes_K K_n)\tilde{} \otimes_K K_m) \cong ((B \otimes_K K_m) \otimes_K K_n)\tilde{}$$

Let us write $K_m = K[t]/(t^{m+1})$, $K_n = K[u]/(u^{n+1})$, and use e for the twisted map from K to K_n.

Note that the 'trivial' map

$$\phi : ((B \otimes_K K_n)\tilde{} \otimes_K K_m) \longrightarrow ((B \otimes_K K_m) \otimes_K K_n)\tilde{}$$

that sends

$$(1 \otimes f(t) \otimes g(u)) \mapsto (1 \otimes g(u) \otimes f(t))$$

is not well-defined. For example, for $c \in K$, in $((B \otimes_K K_n)\tilde{} \otimes_K K_m)$,

$$(1 \otimes 1 \otimes c) = (1 \otimes e(c) \otimes 1)$$

yet, in $((B \otimes_K K_m) \otimes_K K_n)\tilde{}$,

$$\phi(1 \otimes 1 \otimes c) \neq \phi(1 \otimes e(c) \otimes 1).$$

But a slight variation of this map does work.

First we claim that any non-zero element of $((B \otimes_K K_n)\tilde{} \otimes_K K_m)$ can be written uniquely as a sum, $\sum_{i\leq m, j\leq n}(b_{ij} \otimes u^j \otimes t^i)$. Clearly, it suffices to prove this for elements of the form $(b \otimes a_1 u^j \otimes a_2 t^i)$. And we see that

$$(b \otimes a_1 u^j \otimes a_2 t^i) = (b \otimes e(a_2) a_1 u^j \otimes t^i) = \sum_{k\leq n} \left(b \otimes D_k(a_2) a_1 u^{j+k} \otimes t^i\right)$$
$$= \sum_{k\leq n} \left(D_k(a_2) a_1 b \otimes u^{j+k} \otimes t^i\right),$$

as desired. Uniqueness is obvious. Next we observe that this also holds in the algebra $((B \otimes_K K_m) \otimes_K K_n)\tilde{}$. Note that $(b \otimes a_1 t^i \otimes a_2 u^j) \in ((B \otimes_K K_m) \otimes_K K_n)\tilde{}$ equals $(a_1 a_2 b \otimes t^i \otimes u^j)$.

Define

$$\theta : ((B \otimes_K K_n)\tilde{} \otimes_K K_m) \longrightarrow ((B \otimes_K K_m) \otimes_K K_n)\tilde{}$$

by $\theta(b \otimes u^j \otimes t^i) = (b \otimes t^i \otimes u^j)$. Clearly, θ is a ring homomorphism and injective, but we need to show that it is K-linear and surjective. (This sounds completely obvious, but the $\tilde{}$'s make this more subtle than it first appears.) Let $c \in K, (b \otimes u^j \otimes t^i) \in ((B \otimes_K K_n)\tilde{} \otimes_K K_m)$. Then

$$c \cdot (b \otimes u^j \otimes t^i) = (b \otimes u^j \otimes ct^i) = \sum_{k\leq n} \left(D_k(c) b \otimes u^{j+k} \otimes t^i\right)$$

and

$$\begin{aligned}\theta\Bigg(\sum_{k\leq n}(D_k(c) b \otimes u^{j+k} \otimes t^i)\Bigg) &= \sum_{k\leq n} \left(D_k(c) b \otimes t^i \otimes u^{j+k}\right) \\ &= \sum_{k\leq n} \left(b \otimes t^i \otimes D_k(c) u^{j+k}\right) \\ &= (b \otimes t^i \otimes e(c) u^j) = c \cdot (b \otimes t^i \otimes u^j).\end{aligned}$$

This proves K-linearity.

To prove that θ is surjective, it will suffice to show that for all $c \in K$, that $(1 \otimes 1 \otimes c) \in ((B \otimes_K K_m) \otimes_K K_n)\tilde{}$ is in the image of θ. The rest then follows easily. By Proposition 1.14, we can rewrite c as $c = \sum_{k\leq n} e(c_k) u^k$, so we get that

$$(1 \otimes 1 \otimes c) = \Bigg(1 \otimes 1 \otimes \sum_{k\leq n} e(c_k) u^k\Bigg) = \sum_{k\leq n} \left(e(c_k) \otimes 1 \otimes u^k\right).$$

Thus $\theta\left(\sum_{k\leq n}(e(c_k) \otimes u^k \otimes 1)\right) = (1 \otimes 1 \otimes c)$. ∎

For completeness, we mention the following, which can be proved in the same way as the previous theorem.

THEOREM 2.7: *Let X be a K-scheme, $m, n \leq \infty$. Then*

$$P_m(P_n(X)) = P_n(P_m(X)).$$

Remark 2.8: Let X be a K-scheme. For $0 \leq m \leq n \leq \infty$, the maps $f_{mn} : \mathrm{HS}^m_{A/(K,\underline{D})} \to \mathrm{HS}^n_{A/(K,\underline{D})}$ of Remark 1.10.(3) give rise to morphisms

$$f_{mn} : \mathrm{HS}^m_{X/(K,\underline{D})} \longrightarrow \mathrm{HS}^n_{X/(K,\underline{D})}$$

which again form a directed system.

In terms of schemes, the f_{mn} give morphisms

$$\pi_{nm} : P_n(X/(K,\underline{D})) \longrightarrow P_m(X/(K,\underline{D}))$$

which also form a directed system. By Remark 1.10.(3),

$$\mathrm{HS}^\infty_{X/(K,\underline{D})} = \varinjlim_{i\in\mathbb{N}} \mathrm{HS}^i_{X/(K,\underline{D})}$$

and

$$P_\infty(X/(K,\underline{D})) = \varprojlim_{i\in\mathbb{N}} P_i(X/(K,\underline{D})).$$

FUNCTORIAL PROPERTIES. There are numerous easy to verify functorial properties of these constructions, exactly as in [Voj06]. We only mention a few here. Some more general results hold.

For all m, $\mathrm{HS}^m_{A/(K,\underline{D})}$ is functorial in pairs $(K,\underline{D}) \to A$, and $\mathrm{HS}^m_{X/(K,\underline{D})}$ and $P_m(X/(K,\underline{D}))$ are functorial in pairs $X \to \operatorname{Spec} K$. Given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ \uparrow & & \uparrow \\ (K,\underline{D}) & \longrightarrow & (K',\underline{D}) \end{array}$$

there is an induced commutative diagram

$$\begin{array}{ccc} \mathrm{HS}^m_{A/(K,\underline{D})} & \xrightarrow{\mathrm{HS}^m_\phi} & \mathrm{HS}^m_{A'/(K',\underline{D})} \\ \phi \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

that takes $d_i a \in \mathrm{HS}^m_{A/(K,\underline{D})}$ to $d_i\phi(a) \in \mathrm{HS}^m_{A'/(K',\underline{D})}$ for all $a \in A$ and all $i \leq m$.

Two important cases are base change in $(K, \underline{D})$, and functoriality in A, when $K = K'$. One also has the following easy lemma.

Lemma 2.9: *Let A be a $(K, \underline{D})$-algebra, $(K', \underline{D})$ a $\mathcal{D}$-extension field of K, and $A' = A \otimes_K K'$. Then $\mathrm{HS}^m_{A'/(K',\underline{D})} \cong \mathrm{HS}^m_{A/(K,\underline{D})} \otimes_K K'$ as A'-algebras.*

Proof. Let ϕ be the map from $\mathrm{HS}^m_{A'/(K',\underline{D})}$ to $\mathrm{HS}^m_{A/(K,\underline{D})} \otimes_K K'$ that sends $d_k(a \otimes c), k \leq m, a \in A, c \in K$, to $\sum_{i+j=k}(d_i a \otimes 1)(1 \otimes D_j c))$. It is clear that ϕ is an isomorphism. ∎

These properties carry over to schemes. The next result is an easy corollary of the above lemma.

Corollary 2.10: *Let $(K, \underline{D})$ be a $\mathcal{D}$-field, and let $(K', \underline{D})$ be a $\mathcal{D}$-field extension. Then for all K-schemes X and all m,*

$$P_m(X \times_K \operatorname{Spec} K') \cong P_m(X) \times_K \operatorname{Spec} K'.$$

If $f : X \to X'$ is a morphism of K-schemes, one has the following commutative diagram, which lifts f to a map between prolongations.

$$\begin{array}{ccc} P_m(X) & \xrightarrow{P_m(f)} & P_m(X') \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad f \quad} & X' \end{array}$$

Lemma 2.11: *Let X, X' be K-schemes, and $f : X \to X'$ a closed immersion. Then $P_m(f) : P_m(X) \to P_m(X')$ is also a closed immersion.*

Proof. It is enough to check locally, on affines, where it follows from Proposition 1.22. ∎

The following propositions are versions of standard facts about jet spaces, and can be proved in the same way (for example, see [Bli05]).

Proposition 2.12: *Let $f : X \to Y$ be an étale morphism of schemes over a $\mathcal{D}$-field $(K, \underline{D})$. Then for all m,*

$$P_m(X) \cong X \times_Y P_m(Y).$$

Proof. We argue on the corresponding functor of points. For any K-scheme Z,

$$\operatorname{Hom}_K(Z, P_m(X)) \simeq \operatorname{Hom}_K((Z \times_K \operatorname{Spec} K_m)^{\sim}, X)$$

$$\begin{aligned}\operatorname{Hom}_K(Z, X \times_Y P_m(Y)) \\ &\simeq \operatorname{Hom}_K(Z, X) \times_{\operatorname{Hom}_K(Z,Y)} \operatorname{Hom}_K(Z, P_m(Y)) \\ &\simeq \operatorname{Hom}_K(Z, X) \times_{\operatorname{Hom}_K(Z,Y)} \operatorname{Hom}_K((Z \times_K \operatorname{Spec} K_m)^{\sim}, Y)\end{aligned}$$

Consider the following diagram.

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \phi\uparrow & & \uparrow\psi \\ Z & \longrightarrow & (Z \times_K \operatorname{Spec} K_m)^{\sim} \end{array}$$

(with a diagonal arrow τ from $(Z \times_K \operatorname{Spec} K_m)^{\sim}$ to X). A morphism τ is a Z-valued point of $P_m(X)$, and determines a pair of morphisms

$$(\phi, \psi) \in \operatorname{Hom}_K(Z, X) \times_{\operatorname{Hom}_K(Z,Y)} \operatorname{Hom}_K((Z \times_K \operatorname{Spec} K_m)^{\sim}, Y),$$

which determines a Z-valued point of $X \times_Y P_m(Y)$. This gives the canonical map from $P_m(X)$ to $X \times_Y P_m(Y)$, which does not depend on any properties of the morphism f.

In the other direction, a Z-valued point of $X \times_Y P_m(Y)$ corresponds to a pair of morphisms (ϕ, ψ) making the above diagram commute. By formal étaleness, there is a unique τ completing the diagram. Thus, the map taking (ϕ, ψ) to τ determines the inverse morphism from $X \times_Y P_m(Y)$ to $P_m(X)$, as desired. ∎

PROPOSITION 2.13: *Let X be a smooth scheme over the $\mathcal{D}$-field $(K, \underline{D})$ of dimension n. Then for all $m \in \mathbb{N}$, $P_m(X)$ is an $\mathbb{A}^{nm}$-bundle over X. (That is, X can be covered by open sets U such that $P_m(U) \cong U \times_K \mathbb{A}^{nm}$.)*

Proof. By hypothesis, $X \to \operatorname{Spec} K$ is a smooth map, so, by [EGA], this implies that there is a covering of X by open sets U_i, such that for all i, the following diagram commutes

$$\begin{array}{ccc} U_i & \xrightarrow{g_i} & \mathbb{A}^n \\ \downarrow & & \downarrow \\ K & \xrightarrow{=} & K \end{array}$$

and g_i is étale. By the previous proposition, $P_m(U_i) \cong U_i \times \mathbb{A}^{nm}$, as desired. ∎

By the same argument, one also gets the following.

COROLLARY 2.14: *Let X be an n-dimensional smooth scheme over the $\mathcal{D}$-field $(K, \underline{D})$. Then for all m, $P_{m+1}(X)$ is an $\mathbb{A}^n$-bundle over $P_m(X)$.*

3. $\mathcal{D}$-Schemes

Many of the definitions and results in this section are from [Bui93].

Definition 3.1: Let $(K, \underline{D})$ be a $\mathcal{D}$-field. A **$\mathcal{D}$-scheme** over $(K, \underline{D})$ is a K-scheme X such that $\mathcal{O}_X$ is a structure sheaf of $\mathcal{D}$-$(K, \underline{D})$-algebras. A **morphism** of $\mathcal{D}$-schemes is a morphism of K-schemes such that the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves of $\mathcal{D}$-$(K, \underline{D})$-algebras.

Example 3.2: Let X be a K-scheme. Then $P_\infty(X)$ is a $\mathcal{D}$-scheme. Given a morphism $f : X \to Y$ of K-schemes, the induced map $f_\infty : P_\infty(X) \to P_\infty(Y)$ is a morphism of $\mathcal{D}$-schemes.

PROPOSITION 3.3: *Let $(A, \underline{D})$ be an $\mathcal{D}$-$(K, \underline{D})$-algebra. There exists a $\mathcal{D}$-scheme $X = \mathcal{D}\text{-Spec}(A, \underline{D})$ such that, forgetting the $\mathcal{D}$-structure on X, X is isomorphic to* Spec A.

Proof. To show that one can add a $\mathcal{D}$-structure to Spec A, it suffices to show that the localization of a $\mathcal{D}$-ring is itself a $\mathcal{D}$-ring. This is the content of Lemma 1.7. ∎

One also has the following $\mathcal{D}$-version of a well-known fact from algebraic geometry. (See [Har77], II. Ex. 2.4 and Prop. II.2.3, or [EH00] Thm. I-40.)

PROPOSITION 3.4: *Let $(A, \underline{D})$ be a $\mathcal{D}$-ring, and $(X, \mathcal{O}_X)$ a* $\mathcal{D}$-scheme. *Then there is a bijection:*

$$\chi : \mathrm{Hom}_{\mathcal{D}-\mathrm{Sch}}(X, \mathrm{Spec}\, A) \longrightarrow \mathrm{Hom}_{\mathcal{D}-\mathrm{Ring}}(A, \Gamma(X, \mathcal{O}_X)).$$

Proof. In the usual case, given a morphism $f : X \to \mathrm{Spec}\, A$, and the associated map $f^\# : \mathcal{O}_{\mathrm{Spec}\, A} \to f_*\mathcal{O}_X$, one gets a homomorphism $A \to \Gamma(X, \mathcal{O}_X)$ by taking global sections. This gives a bijection

$$\chi : \mathrm{Hom}_{\mathrm{Sch}}(X, \mathrm{Spec}\, A) \longrightarrow \mathrm{Hom}_{\mathrm{Rings}}(A, \Gamma(X, \mathcal{O}_X)).$$

By definition, if $f : X \to \mathcal{D}\text{-Spec}\, A$ is a $\mathcal{D}$-morphism, then the induced homomorphism $A \longrightarrow \Gamma(X, \mathcal{O}_X)$ is a homomorphism of $\mathcal{D}$-rings, so one has an injection:

$$\chi_{\mathcal{D}} : \mathrm{Hom}_{\mathcal{D}-\mathrm{Sch}}(X, \mathcal{D}\text{-Spec}\, A) \longrightarrow \mathrm{Hom}_{\mathcal{D}-\mathrm{Ring}}(A, \Gamma(X, \mathcal{O}_X)).$$

To verify surjectivity, it suffices to look carefully at the construction of χ^{-1} in [EH]. ∎

Remark 3.5: Let $X \subseteq \mathbb{A}^n$ be an affine K-scheme,

$$\Gamma(X, \mathcal{O}_X) = K[x_i]_{i=1,\dots,n}/(f_j)_{j\in J}.$$

For all m, $P_m(X)$ is the closed subscheme of $\mathbb{A}^{nm} = \mathrm{Spec}(K[d_k x_i]_{i=1,\dots,n,k=0,\dots,m})$ with

$$\Gamma(P_m(X), \mathcal{O}_{P_m(X)}) = K[d_k x_i]_{i=1,\dots,n,k=0,\dots,m}/(d_k f_j)_{j\in J,k=0,\dots,m}.$$

(This follows from Proposition 1.24.) In particular, for every closed point $(a_1, \dots, a_n) \in X$, the point $(D_k a_i)_{i=1,\dots,n,k=0,\dots,m}$ is in $P_m(X)$. The canonical projection from $P_m(X)$ to X maps a closed point $(a_{ik})_{i=1,\dots,n,k=0,\dots,m}$ to its first n coordinates, $(a_{i0})_{i=1,\dots,n}$.

Next, we define $\mathcal{D}$-polynomial maps between schemes, which we use to define a section of the canonical map $\pi_m : P_m(X) \to X$.

PROPOSITION 3.6: *Let $(K, \underline{D})$ be a $\mathcal{D}$-field. The prolongation functor, that takes a K-scheme X to the $\mathcal{D}$-scheme $P_\infty(X)$, is the right adjoint to the forgetful functor $Y \mapsto Y^!$ from $\mathcal{D}$-schemes to K-schemes.*

Proof. In [Bui93], p. 1405. This also follows easily from Proposition 1.21. ∎

Recall that given a K-scheme X, a K-rational point of X is a K-scheme homomorphism from $\mathrm{Spec}\, K$ to X. Likewise, if X is a $\mathcal{D}$-scheme, we will say that a K-rational point of X is a $\mathcal{D}$-scheme homomorphism from $\mathcal{D}$-$\mathrm{Spec}\, K$ to X. Of course, a $\mathcal{D}$-morphism $f : X \to Y$ naturally induces a map between their K-rational points. The previous proposition immediately implies that there is a natural bijection between K-rational points of X and of $P_\infty(X)$.

Definition 3.7: Let X, Y be K-schemes, and $f : P_\infty(X) \to P_\infty(Y)$ be a $\mathcal{D}$-morphism. The natural bijections

$$\chi : \mathrm{Hom}_K(\mathrm{Spec}\, K, X) \longrightarrow \mathrm{Hom}_{(K,\underline{D})}(\mathcal{D}\text{-Spec}\, K, P_\infty(X))$$

and

$$\zeta : \mathrm{Hom}_K(\mathrm{Spec}\, K, Y) \longrightarrow \mathrm{Hom}_{(K,\underline{D})}(\mathcal{D}\text{-Spec}\, K, P_\infty(Y))$$

and the induced map

$$\hat{f} : \mathrm{Hom}_{(K,\underline{D})}(\mathcal{D}\text{-}\mathrm{Spec}\, K, P_\infty(X)) \longrightarrow \mathrm{Hom}_{(K,\underline{D})}(\mathcal{D}\text{-}\mathrm{Spec}\, K, P_\infty(Y))$$

determine a (set theoretic) map from K-rational points of X to those of Y, given by $\zeta^{-1} \circ \hat{f} \circ \chi$.

A **$\mathcal{D}$-polynomial map** from X to Y is a map on K-rational points of the form $\zeta^{-1} \circ \hat{f} \circ \chi$, for some $\mathcal{D}$-morphism $f : P_\infty(X) \to P_\infty(Y)$.

Schemes X and Y are **$\mathcal{D}$-polynomially isomorphic** if there are $\mathcal{D}$-polynomial maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$.

Remark 3.8: Let $X = \mathrm{Spec}(K[x_i]_{i \le n}/(f_j)_{j \in J})$, so that

$$P_\infty(X) = \mathrm{Spec}(K[d_k x_i]_{i \le n, k < \infty}/(d_k f_j)_{j \in J, k < \infty}).$$

The bijection χ takes $h \in \mathrm{Hom}_K(\mathrm{Spec}\, K, X)$, which is determined by $b_i = h(x_i)$, $i \le n$, to $H \in \mathrm{Hom}_{(K,\underline{D})}(\mathrm{Spec}\, K, P_\infty(X))$ determined by $D_k b_i = H(d_k x_i)$, $i \le n$.

Proposition 3.9: *Let X be a K-scheme, and $m < \infty$. There exists a $\mathcal{D}$-polynomial map $\nabla_m : X \to P_m(X)$ that is a section of the canonical projection $p_m : P_m(X) \to X$.*

Let $f : X \to Y$ be a morphism of K-schemes. Considering f and $P_m(f)$ as maps on K-rational points, the following diagram commutes.

$$\begin{array}{ccc} P_m(X) & \xrightarrow{P_m(f)} & P_m(Y) \\ {\scriptstyle \nabla_m} \uparrow & & \uparrow {\scriptstyle \nabla_m} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

Proof. By the adjointness of $P_\infty(-)$ and $(-)^!$, there is a natural bijection

$$\mathrm{Hom}_K((P_\infty(X))^!, P_m(X)) \simeq \mathrm{Hom}_{(K,\underline{D})}(P_\infty(X), P_\infty(P_m(X))).$$

Let $f : P_\infty(X) \to P_\infty(P_m(X))$ be the $\mathcal{D}$-morphism corresponding to the canonical projection from $(P_\infty(X))^!$ to $P_m(X)$, and let ∇_m be the $\mathcal{D}$-polynomial map corresponding to f. We show that ∇_m has the desired properties.

It suffices to check locally, so suppose that X is given as $\mathrm{Spec}(K[x_i]_{i\leq n}(f_j)_{j\in J})$. By Remark 3.5,

$$P_m(X) = \mathrm{Spec}(K[d_k x_i]_{k\leq m, i\leq n}/(d_k f_j)_{k\leq m, j\in J})$$
$$P_\infty(X) = \mathrm{Spec}(K[d_k x_i]_{k<\infty, i\leq n}/(d_k f_j)_{k<\infty, j\in J})$$
$$P_\infty(P_m(X)) = \mathrm{Spec}(K[d_l d_k x_i]_{i\leq n, k\leq m, l<\infty}/(g_h)_{h\in H})$$

where $(g_h)_{h\in H}$ is the ideal generated by $(d_l d_k f_j)_{j\in J, k\leq m, l<\infty}$. The $\mathcal{D}$-morphism from $P_\infty(X)$ to $P_\infty(P_m(X))$, corresponding to the projection morphism from $P_\infty(X)$ to $P_m(X)$ is determined by the $\mathcal{D}$-algebra homomorphism

$$K[d_l d_k x_i]_{i\leq n, k\leq m, l<\infty}/(g_h)_{h\in H} \longrightarrow K[d_k x_i]_{k<\infty, i\leq n}/(d_k f_j)_{k<\infty, j\in J}$$

that sends $d_l d_k x_i$ to $\binom{k+l}{k} d_{k+l} x_i$. One can then see that this determines the $\mathcal{D}$-polynomial map from X to $P_m(X)$ that takes the closed point $(a_i)_{i\leq n}$ to $(D_k a_i)_{i\leq n, k\leq m}$. By Remark 3.5, this is a section of π_m.

Next we argue that $P_m(f)\circ\nabla_X = \nabla_Y\circ f$. Again, it suffices to prove this for affine schemes, so assume that $X = \mathrm{Spec}\, K[\overline{x}]/I$ and $Y = \mathrm{Spec}\, K[\overline{y}]/J$. Let $S = K[\overline{x}]/I$ and $R = K[\overline{y}]/J$, and let f also denote the homomorphism from R to S corresponding to $f : X \to Y$. A K-rational point of X corresponds to a homomorphism g from S to K, which is determined by the image of $\overline{x}$, so we think of a K-rational point as a tuple $\overline{a} = g(\overline{x})$ of elements of K. Also $P_m(X) = \mathrm{Spec}\,\mathrm{HS}^m_{S/(K,\underline{D})}$ is affine, and $\mathrm{HS}^m_{S/(K,\underline{D})}$ is generated by $(d_k x)_{x\in\overline{x}, k\leq m}$. We saw above that

$$\nabla_X(\overline{a}) = (\overline{a}, D_1(\overline{a}), \ldots, D_m(\overline{a})).$$

More precisely, $\nabla_X(\overline{a})$ is the K-rational point of $P_m(X)$ that corresponds to the map that sends $d_k x \in \mathrm{HS}^m_{S/(K,\underline{D})}$ to $D_k(g(x)) \in K$, for $x \in \overline{x}$.

Let $f(\overline{a}) = \overline{b} \in Y$, $\overline{b} = (g\circ f(y))_{y\in\overline{y}}$. As above, $\nabla_Y(\overline{b}) = (\overline{b}, D_1(\overline{b}), \ldots, D_m(\overline{b}))$. As a map of K-algebras, $P_m(f)$ sends $d_k y$ to $d_k f(y)$, for $y \in \overline{y}, k \leq m$. Thus,

$$P_m(f)(\overline{a}, D_1(\overline{a}), \ldots, D_m(\overline{a})) = (\overline{b}, D_1(\overline{b}), \ldots, D_m(\overline{b}))$$

as desired. ∎

The following result is new. It generalizes the well-known fact that the first prolongation of a variety is differentially isomorphic to the tangent space. The standard proof is geometric, using the existence, for any variety X, of a differential section $\nabla : X \to P_1(X)$, and fact that $P_1(X)$ is a TX-torsor. In contrast, our proof below is completely algebraic, though in remarks after the proof we try to provide some geometric intuition. (Recall that, in general for $m > 1$, $J_m(X)$ is not a group scheme over X, so $P_m(X)$ is not a torsor under $J_m(X)$. Thus one cannot generalize the standard proof.)

THEOREM 3.10: *Let X be a K-scheme.*

(1) *$P_\infty(P_m(X))$ and $P_\infty(J_m(X))$ are isomorphic as $\mathcal{D}$-schemes.*
(2) *$P_m(X)$ and $J_m(X)$ are $\mathcal{D}$-polynomially isomorphic.*

Proof. Part (2) follows immediately from (1), so it suffices to prove (1). We first establish this for affine schemes. The general argument follows by gluing.

Let $X = \operatorname{Spec} A$. Note that

$$P_\infty(P_m(X)) = \operatorname{Spec}\left(\mathrm{HS}^\infty_{\mathrm{HS}^m_{A/(K,\underline{D})}/(K,\underline{D})}\right)$$

and

$$P_\infty(J_m(X)) = \operatorname{Spec}\left(\mathrm{HS}^\infty_{\mathrm{HS}^m_{A/K}/(K,\underline{D})}\right).$$

Thus we must show that $\mathrm{HS}^\infty_{\mathrm{HS}^m_{A/(K,\underline{D})}/(K,\underline{D})}$ and $\mathrm{HS}^\infty_{\mathrm{HS}^m_{A/K}/(K,\underline{D})}$ are isomorphic as $\mathcal{D}$-$(K,\underline{D})$-algebras. Therefore, the theorem follows from the following

PROPOSITION 3.11: *Let A be a $(K,\underline{D})$-algebra. Then*

$$\left(\mathrm{HS}^\infty_{\mathrm{HS}^m_{A/(K,\underline{D})}/(K,\underline{D})}, \underline{D}\right) \cong \left(\mathrm{HS}^\infty_{\mathrm{HS}^m_{A/K}/(K,\underline{D})}, \underline{D}\right).$$

Proof of Proposition. We first treat the case A a polynomial ring, $A = K[\overline{x}]$, where $\overline{x}$ is a (possibly infinite) tuple. Write

$$R := \mathrm{HS}^\infty_{\mathrm{HS}^m_{A/(K,\underline{D})}/(K,\underline{D})} \cong K[d_i\delta_j x]_{0\le i<\infty, 0\le j\le m, x\in\overline{x}}$$
$$S := \mathrm{HS}^\infty_{\mathrm{HS}^m_{A/K}/(K,\underline{D})} \cong K[d_i\partial_j x]_{0\le i<\infty, 0\le j\le m, x\in\overline{x}}.$$

(Note that $d_i\delta_j x$ and $d_i\partial_j x$ are individual symbols. One could just have well written instead x_{ij}, but the chosen notation is more suggestive. Below, we often write $\delta_j x$, or $\partial_j x$, for $d_0\delta_j x$, or $d_0\partial_j x$, since we are thinking of d_0 as the 'identity map.') Observe that R and S are also $\mathcal{D}$-rings, letting $D_l(d_i\delta_j x) = \binom{i+l}{l} d_{i+l}\delta_j x$ in R, likewise for S. We often write $D_i\partial_j x$, or $D_i\delta_j x$, for $d_i\partial_j x$, or $d_i\delta_j x$.

These rings are obviously isomorphic, but we want to construct an isomorphism that we can also use in the general case, $B = K[\overline{x}]/I$, I any ideal.

Let $\phi : R \to S$ be the K-algebra homomorphism determined by setting

$$\phi(d_i\delta_j x) = D_i\left(\sum_{k+l=j} d_k\partial_l x\right)$$

for all i, j, and x. (Of course, $\phi(c) = c$, for $c \in K$.) Moreover, it is clear that ϕ is actually a $\mathcal{D}$-$(K, \underline{D})$-algebra homomorphism.

To prove that ϕ is an isomorphism, we define a homomorphism $\psi : S \to R$ and show that they are inverses of each other. Let ψ be the homomorphism determined by, for all i, j, and x,

$$\psi(d_i \partial_j x) = D_i \bigg(\sum_{k+l=j} (-1)^k D_k \delta_l x \bigg)$$

As above, one sees easily that ψ is also a $\mathcal{D}$-$(K, \underline{D})$-algebra homomorphism.

First, we show that $\psi \circ \phi = \mathrm{Id}_R$. It suffices to calculate this on the generators, $d_i \delta_j x$.

$$\begin{aligned}
\psi \circ \phi(d_i \delta_j x) &= \psi\bigg(D_i \sum_{k+l=j} d_k \partial_l x \bigg) = D_i \bigg(\sum_{k+l=j} \psi(d_k \partial_l x) \bigg) \\
&= D_i \bigg(\sum_{k+l=j} D_k \bigg(\sum_{a+b=l} (-1)^a D_a \delta_b x \bigg)\bigg) \\
&= D_i(\delta_j x) + D_i \bigg(\sum_{b=0}^{j-1} \bigg(\sum_{a+k=j-b} (-1)^a D_a D_k \delta_b x \bigg)\bigg) \\
&= d_i \delta_j x + D_i \bigg(\sum_{b=0}^{j-1} \bigg(\sum_{a+k=j-b} (-1)^a \tbinom{j-b}{a} D_{j-b} \delta_b x \bigg)\bigg) = d_i \delta_j x
\end{aligned}$$

using the identity $(1-1)^n = \sum_{i=0}^n (-1)^i \binom{n}{i} = 0$, for $n = j - b$. Next, we show that $\phi \circ \psi = \mathrm{Id}_S$, arguing again only on generators.

$$\begin{aligned}
\phi \circ \psi(d_i \partial_j x) &= \phi\bigg(D_i \sum_{k+l=j} (-1)^k D_k \delta_l x \bigg) = D_i \bigg(\sum_{k+l=j} (-1)^k D_k \phi(\delta_l x) \bigg) \\
&= D_i \bigg(\sum_{k+l=j} (-1)^k D_k \sum_{a+b=l} D_a \partial_b x \bigg) \\
&= D_i(\partial_j x) + \bigg(\sum_{b=0}^{j-1} \sum_{k+a=j-b} (-1)^k D_k D_a \partial_b x \bigg) \\
&= d_i \partial_j x + \bigg(\sum_{b=0}^{j-1} \sum_{k=0}^{j-b} (-1)^k \tbinom{j-b}{k} \partial_b x \bigg) = d_i \partial_j x
\end{aligned}$$

This completes the proof for A a polynomial ring. We now consider the general case B a K-algebra, $B = K[\overline{x}]/I$, I an ideal. By Corollary 1.24

and the analogous result in Vojta, one gets the following description of $U := \mathrm{HS}^{\infty}_{\mathrm{HS}^m_{B/(K,\underline{D})}/(K,\underline{D})}$. (We change the notation slightly, adding d_0 and ∂_0 as 'identity functions.')

$$U \cong K[d_i\delta_j x]_{0\leq i, 0\leq j\leq m, x\in\overline{x}}/(d_i\delta_j f)_{0\leq i,0\leq j\leq m, f\in I}$$

For a polynomial $h \in K[\overline{x}]$, the expression $d_i\delta_j h$ should be considered shorthand for an element of the polynomial ring $K[d_i\delta_j x]_{0\leq i,0\leq j\leq m,x\in\overline{x}}$, that can be specified inductively as follows.

$$d_i\delta_j c = \tbinom{i+j}{i} D_{i+j}(c) \quad \text{for } c \in K;$$

$$d_i\delta_j(f+g) = d_i\delta_j f + d_i\delta_j g;$$

$$d_i\delta_j(fg) = \textstyle\sum_{s+t=i}\Big(\textstyle\sum_{k+l=j}(d_s\delta_k f)(d_t\delta_l g)\Big).$$

Likewise, we get the following description of $V := \mathrm{HS}^{\infty}_{\mathrm{HS}^m_{B/K}/(K,\underline{D})}$.

$$V \cong K[d_i\partial_j x]_{0\leq i,0\leq j\leq m,x\in\overline{x}}/(d_i\partial_j f)_{0\leq i,0\leq j\leq m,f\in I}$$

In V, $d_i\delta_j h$ is defined as in U, except that for $c \in K$, $d_i\partial_j c = 0$ for $j > 0$, and $d_i\partial_j c = D_i c$ for $j = 0$.

Rings U and V are quotients of R and S, defined above, $U = R/J$ and $V = S/L$, where J and L are the ideals from the definitions of U and V. We claim that the isomorphism ϕ from R to S naturally induces an isomorphism from U to V. To prove this, it will suffice to show that $\phi(J) \subseteq L$ and $\psi(L) \subseteq J$. This follows immediately from the next two claims.

Claim 1. 1: For each polynomial $h \in A$ and $d_i\delta_j h \in R$,

$$\phi(d_i\delta_j h) = D_i\bigg(\sum_{k+l=j} d_k\partial_l h\bigg).$$

Claim 2. 1: For each polynomial $h \in A$ and $d_i\partial_j h \in S$,

$$\psi(d_i\partial_j h) = D_i\bigg(\sum_{k+l=j}(-1)^k D_k\delta_l h\bigg).$$

Proof of Claim 1. Since all maps being considered are additive, it suffices to consider the case h a monomial, $h = a\overline{y}$, $a \in K$ and $\overline{y} = (y_1, \ldots, y_n)$. We introduce the following multi-index notation. A multi-index α, $(\beta, \gamma$, etc.) is a sequence of non-negative integers, $\alpha = (\alpha_1, \ldots, \alpha_n)$. We say that the **length** of α is n, and write $\sum \alpha$ for $\sum_i \alpha_i$.

Using the (generalized) product rule, we can now give an explicit definition, in R, of

$$d_i\delta_j a\overline{y} := D_i\Bigg(\sum_{k+l=j} \delta_k a \delta_l \overline{y}\Bigg) = D_i\Bigg(\sum_{k+l=j} D_k a \cdot \Bigg(\sum_{|\alpha|=n, \sum\alpha=l} \delta_{\alpha_1} y_1 \cdots \delta_{\alpha_n} y_n\Bigg)\Bigg)$$

Likewise, there is an analogous definition for elements of the ring S.

$$d_i\delta_j a\overline{y} := D_i\Bigg(a \cdot \Bigg(\sum_{|\alpha|=n, \sum\alpha=j} \delta_{\alpha_1} y_1 \cdots \delta_{\alpha_n} y_n\Bigg)\Bigg)$$

We now calculate $\phi(d_i\delta_j h)$ and $D_i(\sum_{k+l=j} d_k \partial_l h)$ to show they are equal. Thus

$$\begin{aligned}
\phi\Big(D_i\delta_j a\overline{y}\Big) &= \phi\Bigg(D_i\Bigg(\sum_{k=0}^{j} D_k a \cdot \Bigg(\sum_{|\alpha|=n, \sum\alpha=j-k} \delta_{\alpha_1} y_1 \cdots \delta_{\alpha_n} y_n\Bigg)\Bigg)\Bigg) \\
&= D_i\Bigg(\sum_{k=0}^{j} D_k a \cdot \Bigg(\sum_{|\alpha|=n, \sum\alpha=j-k} \phi\Big(\delta_{\alpha_1} y_1 \cdots \delta_{\alpha_n} y_n\Big)\Bigg)\Bigg) \\
&= D_i\Bigg(\sum_{k=0}^{j} D_k a \cdot \Bigg(\sum_{|\beta|=|\gamma|=n, \sum\beta+\sum\gamma=j-k} d_{\beta_1}\partial_{\gamma_1} y_1 \cdots d_{\beta_n}\partial_{\gamma_n} y_n\Bigg)\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
&D_i\Bigg(\sum_{l+m=j} d_l \partial_m a\overline{y}\Bigg) \\
&\quad = D_i\Bigg(\sum_{l=0}^{j} D_l\Bigg(a \sum_{|\alpha|=n, \sum\alpha=j-l} \partial_{\alpha_1} y_1 \cdots \partial_{\alpha_n} y_n\Bigg)\Bigg) \\
&\quad = D_i\Bigg(\sum_{l=0}^{j} \sum_{k+m=l} D_k a \cdot \Bigg(D_m \sum_{|\alpha|=n, \sum\alpha=j-l} \partial_{\alpha_1} y_1 \cdots \partial_{\alpha_n} y_n\Bigg)\Bigg) \\
&\quad = D_i\Bigg(\sum_{k=0}^{j} D_k a \cdot \Bigg(\sum_{|\beta|=|\gamma|=n, \sum\beta+\sum\gamma=j-k} d_{\beta_1}\partial_{\gamma_1} y_1 \cdots d_{\beta_n}\partial_{\gamma_n} y_n\Bigg)\Bigg).
\end{aligned}$$

This completes the proof of Claim 1. ∎

Proof of Claim 2. Again we can assume that h is a monomial, $h = a\overline{y}$, $a \in K$, and $\overline{y} = (y_1, \ldots, y_n)$. We want to show that $\psi(d_i \partial_j a\overline{y})$ equals

$$D_i \sum_{k+l=j} (-1)^k D_k \delta_l (a\overline{y}).$$

We calculate

$$\begin{aligned}
\psi\Big(d_i \partial_j a\overline{y}\Big) &= aD_i \psi\Big(\partial_j \overline{y}\Big) = aD_i \psi\Big(\sum_{|\alpha|=n, \sum \alpha = j} \delta_{\alpha_1} y_1 \cdots \delta_{\alpha_n} y_n\Big) \\
&= aD_i \Big(\sum_{k+l=j} \sum_{|\beta|=|\gamma|=n, \sum \beta = k \sum \alpha = l} (-1)^k d_{\beta_1} \delta_{\gamma_1} y_1 \cdots d_{\beta_n} \delta_{\gamma_n} y_n\Big) \\
&= aD_i \Big(\sum_{k+l=j} (-1)^k D_k \delta_l \overline{y}\Big)
\end{aligned}$$

and

$$\begin{aligned}
&D_i \Big(\sum_{k+l=j} (-1)^k D_k \delta_l a\overline{y}\Big) \\
=&D_i \Big(\sum_{s+t+u+v=j} (-1)^{s+u} D_s \delta_t a \cdot D_u \delta_v \overline{y}\Big) \\
=&aD_i \Big(\sum_{u+v=j} (-1)^u D_u \delta_v \overline{y}\Big) + D_i \sum_{u+v<j} \Big(\sum_{s+t=j-(u+v)} (-1)^{s+u} D_s \delta_t a \cdot D_u \delta_v \overline{y}\Big) \\
=&aD_i \Big(\sum_{u+v=j} (-1)^u D_u \delta_v \overline{y}\Big) \\
&+ D_i \sum_{u+v<j} \Big((-1)^u \sum_{s+t=j-(u+v)} (-1)^s \tbinom{s+t}{s} D_{s+t} a \cdot D_u \delta_v \overline{y}\Big) \\
=&aD_i \Big(\sum_{u+v=j} (-1)^u D_u \delta_v \overline{y}\Big) \\
&+ D_i \sum_{u+v<j} \Big((-1)^u \sum_{s=0}^{j-(u+v)} (-1)^s \tbinom{j-(u+v)}{s} D_{j-(u+v)} a \cdot D_u \delta_v \overline{y}\Big) \\
=&aD_i \Big(\sum_{u+v=j} (-1)^u D_u \delta_v \overline{y}\Big).
\end{aligned}$$

This completes the proof of Claim 2. ∎

This also completes the proof of the Proposition. ∎

The proof of the Theorem is completed. ∎

Remark 3.12: Our original definition of ψ was

$$\psi(d_i\partial_j x) = D_i\Bigg(\sum_{k+l=j}\sum_{\pi\in P[k]}(-i)^{|\pi|}D_\pi\delta_l x\Bigg)$$

where $P[k]$ is the set of ordered partitions π of k, that is, $\pi = (a_1,\ldots,a_n) \in (\mathbb{N}^+)^n$, with $\sum_{p=1}^n a_p = k$ and $D_\pi = D_{a_1}\circ\cdots\circ D_{a_n}$. The length of π is denoted $|\pi|$. This is the formula one finds if one inverts ϕ 'by hand' on examples with i, j small. Later, we observed that the following lemma yields the definition that we gave above,

$$\psi(d_i\partial_j x) = D_i\Bigg(\sum_{k+l=j}(-1)^k D_k\delta_l x\Bigg).$$

Definition 3.13: Given $k \in \mathbb{N}^+$, define a function, 'multinomial', $\mu : P[k] \to \mathbb{N}^+$, by

$$\mu(a_1,\ldots,a_n) = \tbinom{k}{a_1,\ldots,a_n} := \frac{k!}{a_1!\cdots a_n!}.$$

One can easily check that $D_\pi = \mu(\pi)D_k$.

Lemma 3.14: *For all $k \in \mathbb{N}^+$,*

$$\sum_{\pi\in P[k]}(-1)^{|\pi|}\mu(\pi) = (-1)^k.$$

Proof. By induction on k. It will be helpful to stipulate that $P[0] := \{\emptyset\}$, $|\emptyset| = 0$, and $\mu(\emptyset) = 1$. The case $k = 1$ is obvious, so assume the lemma holds up to $k-1$.

Given a partition $\pi \in P[j]$, $\pi = (a_1,\ldots,a_m)$, and $i \in \mathbb{N}^+$, let $\pi * (i)$ denote the partition $(a_1,\ldots,a_m,i) \in P[j+i]$. Note that $\mu(\pi*(i)) = \binom{j+i}{i}\mu(\pi)$.

$$\begin{aligned}\sum_{\pi\in P[k]}(-1)^{|\pi|} &= \sum_{i=1}^{k}\sum_{\pi\in P[k-i]}(-1)^{|\pi|+1}\mu(\pi*(i)) = -\sum_{i=1}^{k}\tbinom{k}{i}\sum_{\pi\in P[k-i]}(-1)^{|\pi|}\mu(\pi)\\ &= -\sum_{i=1}^{k}\tbinom{k}{i}(-1)^{k-i} = (-1)^{k+1}\sum_{i=1}^{k}\tbinom{k}{i}(-1)^i\\ &= (-1)^k\end{aligned}$$ ∎

Remark 3.15: We now explain the geometric intuition behind the proof of the preceding proposition. Recall that by the characterizations of jet spaces and prolongations via representable functors, we have the following natural bijections,

$$\mathrm{Hom}_K(\operatorname{Spec} K, J_m(X)) \simeq \mathrm{Hom}_K(\operatorname{Spec} K_m, X)$$
$$\mathrm{Hom}_K(\operatorname{Spec} K, P_m(X)) \simeq \mathrm{Hom}_K(\operatorname{Spec} \tilde{K}_m, X)$$

where, for example, $\mathrm{Hom}_K(\operatorname{Spec} K, J_m(X))$ is the set of K-rational points of $J_m(X)$. By Proposition 1.14, there is an isomorphism Ψ from K_m to $\tilde{K}_m$ so there is a corresponding isomorphism Ψ' from $\operatorname{Spec} \tilde{K}_m$ to $\operatorname{Spec} K_m$. Thus, Ψ' induces a natural bijection from $\mathrm{Hom}_K(\operatorname{Spec} K_m, X)$ to $\mathrm{Hom}_K(\operatorname{Spec} \tilde{K}_m, X)$, and thus between the K-points of $J_m(X)$ and $P_m(X)$. When one computes this map in local coordinates, one gets the morphisms from the proof of the preceding theorem.

We illustrate this for $X = \mathbb{A}^1 = \operatorname{Spec} K[x]$. First, we reformulate everything in terms of K-algebras. We have $J_m(X) = \operatorname{Spec} K[\partial_0 x, \ldots, \partial_m x]$ and $P_m(X) = \operatorname{Spec} K[\delta_0 x, \ldots, \delta_m x]$ and the following bijections.

$$\mathrm{Hom}_K(K[\partial_0 x, \ldots, \partial_m x], K) \simeq \mathrm{Hom}_K(K[x], K_m)$$
$$\mathrm{Hom}_K(K[\delta_0 x, \ldots, \delta_m x], K) \simeq \mathrm{Hom}_K(K[x], \tilde{K}_m)$$

An m-jet, $(a_0, \ldots, a_m) \in J_m(X)$, corresponds to the map

$$f : x \mapsto a_0 + a_1 t + \cdots + a_m t^m \in \mathrm{Hom}_K(K[x], K_m),$$

which corresponds to the map $F \in \mathrm{Hom}_K(K[\partial_0 x, \ldots, \partial_m x], K)$, where $F(\partial_i x) = a_i$. Composing f with the isomorphism $\Psi : K_m \to \tilde{K}_m$, one gets the map

$$g : x \mapsto a_0 + (D_0 a_1 + D_1 a_0)t + \cdots + \Bigg(\sum_{j+k=m} D_j a_k\Bigg) t^m \in \mathrm{Hom}_K(K[x], \tilde{K}_m),$$

which corresponds to $G \in \mathrm{Hom}_K(K[\delta_0 x, \ldots, \delta_m x], K$, where

$$G(\delta_i x) = \sum_{j+k=i} D_j a_k.$$

Thus, the bijection above between K-points of $J_m(X)$ and $P_m(X)$ sends $(a_0, \ldots, a_m) \in J_m(X)$ to $(a_0, D_0 a_1 + D_1 a_0, \ldots, \sum_{j+k=m} D_j a_k) \in P_m(X)$. This 'differential map' from $J_m(X)$ to $P_m(X)$ corresponds to the algebraic morphism from $P_\infty(J_m(X))$ to $P_\infty(P_m(X))$ given in the proof of the above theorem by the map ϕ.

4. Multiple derivations

We now develop the theory of prolongations over a differential field with finitely many commuting derivations. In characteristic $p > 0$, Okugawa [Oku87] developed differential algebra over fields with commuting higher derivations. More recently, differential Galois theory for such fields has been investigated by Matzat and van der Put [MvdP03]. Ziegler [Zie03] has shown that the model completion of the theory of n commuting Hasse–Schmidt derivations is a definitional expansion of the theory $\mathrm{SCF}_{p,n}$, the theory of separably closed fields of characteristic p and degree of imperfection n. Kolchin [Kol73] considers differential fields with commuting derivations, mostly of characteristic 0. Moosa, Pillay, and Scanlon [MPS07] study the model theory of characteristic 0 differential fields with n commuting derivations. Since we will consider rings (fields) of arbitrary characteristic, the results in this section essentially apply to all of the above contexts.

Of course, it would have been possible to consider multiple derivations from the beginning. But it is easier to see the theory developed for one derivation first. The general theory is then quite similar.

Definition 4.1: For $n \in \mathbb{N}^+$, a ring with n commuting (higher) derivations is a ring R and a sequence, $\underline{D}_1, \ldots, \underline{D}_n$, of iterative derivations on R, $\underline{D}_i = (D_{i,0}, D_{i,1}, \ldots)$, for $i \leq n$, such that for all i, j, k, l, $D_{i,k} \circ D_{j,l} = D_{j,l} \circ D_{i,k}$.

We also add symbols for 'mixed' derivatives. An n-multi-index α is a sequence $(\alpha_1, \ldots, \alpha_n)$ of non-negative integers. For each n-multi-index α, we add an operator D_α, such that $D_\alpha = D_{1,\alpha_1} \circ \cdots \circ D_{n,\alpha_n}$. So $D_{i,j} = D_\alpha$, where α is the multi-index with a j in the i-th place, and 0's everywhere else.

A ring with n commuting derivations will be written $(R, \underline{D})$ when there no chance of confusion. These will also be called $\mathcal{D}$-rings.

Given an n-multi-index α, the **size** of α, written $|\alpha|$, is the sum $\alpha_1 + \cdots + \alpha_n$. We will sometimes write $\alpha \leq m$ for $|\alpha| \leq m$. There is also a natural partial order on n-multi-indices, where $\alpha \leq \beta$ if and only if for all $i \leq n, \alpha_i \leq \beta_i$. We also writes $\overline{0}$ for the multi-index that is a sequence of 0's. Note that $D_{\overline{0}} = \mathrm{Id}_R$.

Composition of mixed derivatives is completely determined by the iteration rule for each derivation, and the fact that the derivations commute.

LEMMA 4.2: *Let $(R, \underline{D})$ be a $\mathcal{D}$-ring, and let D_α, D_β mixed derivatives, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$.*

$$D_\alpha \circ D_\beta = \binom{\alpha_1+\beta_1}{\alpha_1} \cdots \binom{\alpha_n+\beta_n}{\alpha_n} D_{\alpha+\beta}$$

One also has the following generalization of the Leibniz Rule.

PROPOSITION 4.3: *Let $(R, \underline{D})$ be a $\mathcal{D}$-ring. Then for any multi-index α and any $a, b \in R$,*

$$D_\alpha(ab) = \sum_{\beta+\gamma=\alpha} D_\beta(a) D_\gamma(b)$$

Proof.

$$\begin{aligned}
&D_\alpha(ab)\\
&= D_{1,\alpha_1} \circ \cdots \circ D_{n,\alpha_n}(ab)\\
&= D_{1,\alpha_1} \circ \cdots \circ D_{n-1,\alpha_{n-1}}\left(\sum_{\beta_n+\gamma_n=\alpha_n} D_{n,\beta_n}(a) D_{n,\gamma_n}(b)\right)\\
&= \sum_{\beta_n+\gamma_n=\alpha_n} \left(D_{1,\alpha_1} \circ \cdots \circ D_{n,\alpha_n}\left(D_{n,\beta_n}(a) D_{n,\gamma_n}(b)\right)\right)\\
&\vdots\\
&= \sum_{\beta_1+\gamma_1=\alpha_1} \left(\cdots\left(\sum_{\beta_n+\gamma_n=\alpha_n} \left(D_{1,\beta_1} \circ \cdots \circ D_{n,\beta_n}(a) \cdot D_{1,\gamma_1} \circ \cdots \circ D_{n,\gamma_n}(b)\right)\right)\right)\\
&= \sum_{\beta+\gamma=\alpha} D_\beta(a) D_\gamma(b)
\end{aligned}$$

Remark 4.4: One sees this rule, for example, when taking Taylor series of holomorphic functions of n variables. Given functions f and g, the Leibniz Rule computes the coefficent of z^α in the Taylor series of fg from the coefficients of the Taylor series of f and of g.

Commuting derivations behave well under localization.

LEMMA 4.5: *let $(R, \underline{D})$ be a $\mathcal{D}$-ring, with n commuting derivations, and $S \subseteq R$ a multiplicative subset. Then the unique extensions of each of the derivations on R to $S^{-1}R$ also commute.*

Proof. See [Oku87], Section 1.6, Corollary 1. ∎

Definition 4.6: Let $(R, \underline{D})$ be a $\mathcal{D}$-ring. Given $(R, \underline{D})$-algebras $f : R \to A$ and B, a **higher derivation of order** m from A to B over $(R, \underline{D})$ is a set of maps $\{D_\alpha : \alpha \leq m\}$ such that $D_{\overline{0}}$ is an R-algebra homomorphism, the D_α are (additive) abelian group homomorphisms, and

(1) $D_\alpha(f(x)) = D_{\overline{0}}(f(D_\alpha(x)))$;
(2) (Leibniz Rule) $D_\alpha(ab) = \sum_{\beta+\gamma=\alpha} D_\beta(a)D_\gamma(b)$, for all $a, b, \in A, \alpha \leq m$.

Definition 4.7: Let $(R, \underline{D})$ be a $\mathcal{D}$-ring, $f : R \to A$ an $(R, \underline{D})$-algebra. Define $\mathrm{HS}^m_{A/(R,\underline{D})}$ to be the A-algebra that is the quotient of the polynomial algebra $A[x^{(\alpha)}]_{x \in A, 0 \neq \alpha \leq m}$ by the ideal I generated by:

(1) $(x + y)^{(\alpha)} - x^{(\alpha)} - y^{(\alpha)} : x, y \in A, 0 \neq \alpha \leq m$;
(2) $(xy)^{(\alpha)} - \sum_{\beta+\gamma=\alpha} x^{(\beta)}y^{(\gamma)} : x, y \in A, 0 \neq \alpha \leq m$;
(3) $f(r)^{(\alpha)} - f(D_\alpha(r)) : r \in R, 0 \neq \alpha \leq m$.

In $A[x^{(\alpha)}]$, we identify $x \in A$ with $x^{(\overline{0})}$. There is a universal derivation $\underline{d} = \{d_\alpha : \alpha \leq m\} : A \to \mathrm{HS}^m_{A/(R,\underline{D})}$ such that for $\alpha \leq m$ and $x \in A, d_\alpha(x) = x^{(\alpha)}$.

Remark 4.8: As before, for $m = \infty$, because $(R, \underline{D})$ is an iterative $\mathcal{D}$-ring, there is a canonical way to make $\mathrm{HS}^\infty_{A/(R,\underline{D})}$ into a $\mathcal{D}$-$(R, \underline{D})$-algebra.

Extend $\underline{d} : A \to \mathrm{HS}^\infty_{A/(R,\underline{D})}$ to an (iterative) higher derivation on $\mathrm{HS}^\infty_{A/(R,\underline{D})}$ by letting

$$d_\alpha(x^{(\beta)}) = \binom{\alpha_1+\beta_1}{\alpha_1} \cdots \binom{\alpha_n+\beta_n}{\alpha_n} x^{(\alpha+\beta)}.$$

Definition 4.9: Let $(R, \underline{D})$ be a $\mathcal{D}$-ring with n commuting derivations. Let

$$R_m = \begin{cases} R[t_1, \ldots, t_n]/(t_1, \ldots, t_n)^{m+1} & \text{for } m < \infty \\ R[[t_1, \ldots, t_n]] & \text{for } m = \infty \end{cases}$$

For a multi-index α, write t^α as shorthand for $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$. For each m, we define the twisted homomorphism $e : R \to R_m$ by

$$e(r) = \sum_{\alpha \leq m} D_\alpha(r)t^\alpha.$$

Let $\tilde{R}^m$ be the R-algebra isomorphic to R_m as a ring, and made into an R-algebra via the map $e : R \to R_m$.

Likewise, given an $(R, \underline{D})$-algebra $f : R \to B$, let

$$B_m = B[t_1, \ldots, t_n]/(t_1, \ldots, t_n)^{m+1},$$

for $m < \infty$, and $B_\infty = B[[t_1, \dots, t_n]]$. Define $\tilde{f} : R \to B_m$ by

$$\tilde{f}(r) = \sum_{\alpha \leq m} f(D_\alpha(r))t^\alpha.$$

and let $\tilde{B}_m$ be the $(R, \underline{D})$-algebra that is the ring B_m with the map $\tilde{f} : R \to B_m$.

That e and $\tilde{f}$ are actually homomorphisms follows immediately from the Leibniz Rule. One also has the following converse, whose proof is immediate.

LEMMA 4.10: *Let R be a ring and let $f : R \to R_m$ be a ring homomorphism, which we write*

$$f(b) = \sum_{\alpha \leq m} f_\alpha(b)t^\alpha.$$

Suppose that $f_{\overline{0}} = \mathrm{Id}_R$. Then the maps $\{f_\alpha : \alpha \leq m\}$ are a higher derivation on R.

PROPOSITION 4.11: *Let $(R, \underline{D})$ be a $\mathcal{D}$-ring with n commuting derivations. For all m, R_m and $\tilde{R}_m$ are isomorphic as R-algebras.*

Proof. The idea of the proof is the same as for Lemma 1.14. We first treat the case $m < \infty$. Let $\psi : R_m \to \tilde{R}_m$ be the map $\psi(r) = e(r) = \sum_{\alpha \leq m} D_\alpha(r)t^\alpha$, for $r \in R$, and $\psi(t_i) = t_i$, for $i \leq m$. This is clearly a homomorphism, so it remains to check that ψ is injective and surjective.

Linearly order the n-multi-indices of size $\leq m$, $\alpha_1, \dots, \alpha_k$ such that, for all $i, j \leq k$, $|\alpha_i| < |\alpha_j|$ implies $i < j$. Let $b \in R_m$ be $b = \sum_{\alpha \leq m} b_\alpha t^\alpha$, and suppose that $\psi(b) = 0$. We will show $b = 0$ by showing that each $b_{\alpha_i} = 0$, by induction on i.

$$\psi(b) = \psi\Bigg(\sum_{\alpha \leq m} b_\alpha t^\alpha\Bigg) = \sum_{\alpha \leq m} \psi(b_\alpha)t^\alpha = \sum_{\alpha \leq m}\Bigg(\sum_{\beta+\gamma=\alpha} D_\beta(b_\gamma)t^\alpha\Bigg)$$

By assumption, each coefficient $\sum_{\beta+\gamma=\alpha} D_\beta(b_\beta)$ of t^α is 0. For the base case, $\alpha_1 = \overline{0}$, the constant term, that is, the coefficient of $t^{\overline{0}}$, is $0 = D_{\overline{0}}(b_{\overline{0}}) = b_{\overline{0}}$.

By induction, suppose that for all $j \leq i$, $b_{\alpha_j} = 0$. The $t^{\alpha_{i+1}}$ coefficient of $\psi(b)$ is $0 = \sum_{\beta+\gamma=\alpha_{i+1}} D_\beta(b_\gamma) = D_{\overline{0}}(b_{\alpha_{i+1}}) = b_{\alpha_{i+1}}$, because for $\beta + \gamma = \alpha_{i+1}$, if $\beta \neq 0$, then $|\gamma| < \alpha_{i+1}$, so $b_\gamma = 0$, by the induction hypothesis.

To show that ψ is surjective, it suffices to show that for each $r \in R$, $r \in \tilde{R}_m$ is in $\mathrm{Im}(\psi)$. For fixed r, we iteratively define a sequence, $c_0, c_1, \dots, c_k$, of elements of R_m with the following properties. One, for all $i \leq k$, the constant term of $\psi(c_i)$, as a polynomial in the t_i, is r. Two, for $i \geq 1$, and $1 \leq j \leq i$, the coefficient of t^{α_j} in $\psi(c_i)$ is 0. Then $\psi(c_m) = r$, as desired. Set $c_0 = r$. For the iterative step, suppose that $c_0, \ldots, c_i$ have been defined, and that $\psi(c_i) = r + \sum_{i+l \le j \le k} a_{\alpha_j} t^{\alpha_j}$. Let $c_{i+1} = c_i - a_{\alpha_{i+1}} t^{\alpha_{i+1}}$. Clearly, this procedure yields such a sequence.

For $m = \infty$, given the isomorphisms $\psi_i : R_i \to \tilde{R}_i$, $i < \infty$, it suffices to note again that R_∞ and $\tilde{R}_\infty$ are the inverse limits of $\{R_i\}_{i<\infty}$ and $\{\tilde{R}_i\}_{i<\infty}$, respectively. The required isomorphism $\psi_\infty : R_\infty \to \tilde{R}_\infty$ also sends $r \in R$ to $e(r)$, and sends each t_i to t_i. ∎

The next two results are proved in the same way as Proposition 1.18 and Lemma 1.19, respectively.

Proposition 4.12: *Let $(R, \underline{D})$ be a $\mathcal{D}$-ring, and $R \to A$ and $R \to B$ be $(R, \underline{D})$-algebras. Given a higher derivation $\underline{\delta} \in \mathrm{Der}^m_{(R,\underline{D})}(A, B)$, there exists a unique $(R, \underline{D})$-algebra homomorphism, $\phi : \mathrm{HS}^m_{A/(R,\underline{D})} \to B$ such that for all $\alpha \le m$, $\delta_\alpha = \phi \circ d_\alpha$. Thus $\mathrm{HS}^m_{A/(R,\underline{D})}$ (together with the universal derivation $\underline{d} : A \to \mathrm{HS}^m_{A/(R,\underline{D})}$) represents the functor $\mathrm{Der}^m_{(R,\underline{D})}(A, -)$.*

Lemma 4.13: *Let $(R, \underline{D}), R \to A, R \to B$, and m be as above. Given $\underline{\delta} \in \mathrm{Der}^m_{(R,\underline{D})}(A, B)$, define a map $\phi = \phi_{\underline{\delta}} : A \to \tilde{B}_m$ by $\phi(a) = \sum_{\alpha \le m} \delta_\alpha(a) t^\alpha$. Then $\phi \in \mathrm{Hom}_R(A, \tilde{B}_m)$ and the map*

$$\underline{\delta} \mapsto \phi_{\underline{\delta}} : \mathrm{Der}^m_{(R,\underline{D})}(A, B) \longrightarrow \mathrm{Hom}_R(A, \tilde{B}_m)$$

is a bijection.

The next corollary is the key result in characterizing prolongations in terms of representable functors, as in Buium.

Corollary 4.14: *There is a natural bijection*

$$\mathrm{Hom}_R(\mathrm{HS}^m_{A/(R,\underline{D})}, B) \longrightarrow \mathrm{Hom}_R(A, \tilde{B}_m).$$

Proof. Immediate from Proposition 4.12 and Lemma 4.13. ∎

The following results are proved as in the case of a single derivation.

Proposition 4.15: *Let $(R, \underline{D})$ be a $\mathcal{D}$-ring, Alg_R be the category of $(R, \underline{D})$-algebras, and $\mathcal{D}$-Alg_R be the category of $\mathcal{D}$-$(R, \underline{D})$-algebras. Let U be the forgetful functor $\mathcal{D}\text{-}\mathrm{Alg}_R \to \mathrm{Alg}_R$. Then the functor $F : \mathrm{Alg}_R \to \mathcal{D}\text{-}\mathrm{Alg}_R$, sending A to $\mathrm{HS}^\infty_{A/(R,\underline{D})}$, is the left adjoint of U.*

PROPOSITION 4.16 (Second fundamental exact sequence): *Let $(R, \underline{D})$ be a $\mathcal{D}$-ring and $R \to A \to B$ a sequence of ring homomorphisms. Assume that $A \to B$ is surjective, and let I be its kernel. Let J be the ideal in $\mathrm{HS}^m_{A/(R,\underline{D})}$ generated by $\{d_\alpha x : \alpha \leq m, x \in I\}$. Then the following sequence is exact.*

$$0 \longrightarrow J \longrightarrow \mathrm{HS}^m_{A/(R,\underline{D})} \longrightarrow \mathrm{HS}^m_{B/(R,\underline{D})} \longrightarrow 0$$

In the definition of J, it suffices to let x vary over a set of generators of I.

PROPOSITION 4.17: *Let $(R, \underline{D})$ be a $\mathcal{D}$-ring, and $A = R[x_i]_{i \in I}$. Then $\mathrm{HS}^m_{A/(R,\underline{D})}$ is the polynomial algebra $A[d_\alpha x_i]_{i \in I, 1 \leq \alpha \leq m}$.*

Remark 4.18: For $m, n \geq 1$, define $c_{n,m}$ to be the number of n-multi-indices of size $\leq m$. Equivalently, $c_{n,m}$ is the number of monomials in n variables of order $\leq m$ or the number of mixed partial derivatives in n variables of total order $\leq m$.

By the previous proposition, given a polynomial ring $A = R[x_1, \ldots, x_q]$ over a ring R with n commuting derivations, then $\mathrm{HS}^m_{A/(R,\underline{D})}$ is a polynomial ring in $q \cdot c_{n,m}$ indeterminates.

COROLLARY 4.19: *Let A be an $(R, \underline{D})$-algebra, $A \cong R[x_i]_{i \in I}/(f_j)_{j \in J}$. Then*

$$\mathrm{HS}^m_{A/(R,\underline{D})} \cong A[d_\alpha x_i]_{i \in I, 1 \leq \alpha \leq m}/(d_\alpha f_j)_{j \in J, 1 \leq \alpha \leq m}.$$

4.1. PROLONGATIONS. In this section, we generalize the results of Section 2 to fields with many derivations. Almost everything goes through as before. Assume throughout that $(K, \underline{D})$ is a $\mathcal{D}$-field with n commuting derivations.

LEMMA 4.20: *Let A be a $(K, \underline{D})$-algebra and S a multiplicative subset of A. There is an isomorphism*

$$\mathrm{HS}^m_{A/(K,\underline{D})} \otimes_A S^{-1}A \longrightarrow \mathrm{HS}^m_{S^{-1}A/(K,\underline{D})}.$$

THEOREM 4.21: *Let X be a K-scheme. For all m, there exists a sheaf of $\mathcal{O}_X$-algebras $\mathrm{HS}^m_{X/(K,\underline{D})}$ such that (i) for each open affine $\operatorname{Spec} A \subseteq X$, there is an isomorphism*

$$\phi_A : \Gamma(\operatorname{Spec} A, \mathrm{HS}^m_{X/(K,\underline{D})}) \longrightarrow \mathrm{HS}^m_{A/(K,\underline{D})}$$

of $(K, \underline{D})$-algebras, and (ii) the various ϕ_A are compatible with the localization isomorphism of Lemma 4.20. Moreover, the collection $((\mathrm{HS}^m_{X/(K,\underline{D})}), (\phi_A)_A)$ is unique.

Definition 4.22: Let X be a K-scheme. For all m, the m-th **prolongation** of X is the scheme

$$P_m(X/(K,\underline{D})) := \mathbf{Spec}\, \mathrm{HS}^m_{X/(K,\underline{D})}.$$

Suppose that A is a $(K,\underline{D})$-algebra. We write

$$P_m(A/(K,\underline{D})) = P_m(\mathrm{Spec}\, A/(K,\underline{D})),$$

which equals $\mathbf{Spec}\, \mathrm{HS}^m_{A/(K,\underline{D})}$.

We will also write X^m or $P_m(X)$ for $P_m(X/(K,\underline{D}))$.

Recall that for $m < \infty$, $K_m = K[t_1,\dots,t_n]/(t_1,\dots,t^{m+1})$, $K_\infty = K[[t_1,\dots,t_n]]$, and that $e : K \to K_m$ denotes the twisted homomorphism. We also let $e : \mathrm{Spec}\, K_m \to \mathrm{Spec}\, K$ denote the corresponding twisted morphism of schemes. Given a K-scheme Y, let $(Y \times_K \mathrm{Spec}\, K_m)\tilde{}$ denote the scheme $(Y \times_K \mathrm{Spec}\, K_m)$ made into a K-scheme via the map $e \circ p : (Y \times_K \mathrm{Spec}\, K_m) \to \mathrm{Spec}\, K$, where $p : (Y \times_K \mathrm{Spec}\, K_m) \to \mathrm{Spec}\, K_m$ is the canonical projection.

Theorem 4.23: *Let X be a K-scheme. For all m, the scheme $P_m(X)$ represents the functor from K-schemes to sets given by*

$$Y \mapsto \mathrm{Hom}_K((Y \times_K \mathrm{Spec}\, K_m)\tilde{}, X).$$

Theorem 4.24 (Moosa, Pillay, and Scanlon): *Let X be a K-scheme. For all $m, q \leq \infty$,*

$$J_m(P_q(X)) \cong P_q(J_m(X)).$$

Proof. Both proofs of Theorem 2.6 generalize easily. Here we only show how to adapt the second proof. Exactly as before, it suffices to show that for any K-algebra B, the following are isomorphic.

$$((B \otimes_K K_q)\tilde{} \otimes_K K_m) \cong ((B \otimes_K K_m) \otimes_K K_q)\tilde{}$$

where $K_m = K[t_1,\dots t_n]/(t_1,\dots,t_n)^{m+1}$, $K_q = K[u_1,\dots,u_n]/(u_1,\dots,u_n)^{q+1}$, and we use e for the twisted map from K to K_q.

We claim any non-zero element of $((B \otimes_K K_q)\tilde{} \otimes_K K_m)$ can be written uniquely as a sum

$$\sum_{\alpha \leq m, \beta \leq q} (b_{\alpha,\beta} \otimes u^\beta \otimes t^\alpha).$$

Again, it suffices to prove this for elements of the form $(b\otimes a_1u^\beta\otimes a_2t^\alpha)$. And

$$(b\otimes a_1u^\beta\otimes a_2t^\alpha)=(b\otimes e(a_2)a_1u^\beta\otimes t^\alpha)=\sum_{\gamma\le q}(b\otimes D_\gamma(a_2)a_1u^{\beta+\gamma}\otimes t^\alpha)$$
$$=\sum_{\gamma\le q}(D_\gamma(a_2)a_1b\otimes u^{\beta+\gamma}\otimes t^\alpha)$$

as desired. Secondly, observe that this also holds in $((B\otimes_K K_m)\otimes_K K_q)$, as $(b\otimes a_1t^\alpha\otimes a_2u^\beta)\in((B\otimes_K K_m)\otimes_K K_q)$ equals $(a_1a_2b\otimes t^\alpha\otimes u^\beta)$.

Define

$$\theta:((B\otimes_K K_q)\tilde{}\otimes_K K_m)\longrightarrow((B\otimes_K K_m)\otimes_K K_q)\tilde{}$$

by $\theta(b\otimes u^\beta\otimes t^\alpha)=(b\otimes t^\alpha\otimes u^\beta)$. It suffices to show that θ is K-linear and surjective. Let $c\in K$, $(b\otimes u^\beta\otimes t^\alpha)\in((B\otimes_K K_q)\tilde{}\otimes_K K_m)$. Then

$$c\cdot(b\otimes u^\beta\otimes t^\alpha)=\sum_{\gamma\le q}(D_\gamma(c)b\otimes u^{\beta+\gamma}\otimes t^\alpha),$$

and

$$\begin{aligned}\theta\Big(\sum_{\gamma\le q}(D\gamma(c)b\otimes u^{\beta+\gamma}\otimes t^\alpha)\Big)&=\sum_{\gamma\le q}(D_\gamma(c)b\otimes t^\alpha\otimes u^{\beta+\gamma})\\&=\sum_{\gamma\le q}(b\otimes t^\alpha\otimes D_\gamma(c)u^{\beta+\gamma})\\&=(b\otimes t^\alpha\otimes e(c)u^\beta)\\&=c\cdot(b\otimes t^\alpha\otimes u^\beta).\end{aligned}$$

This proves K-linearity.

To prove that θ is surjective, it will suffice to show that for all $c\in K$, that $(1\otimes 1\otimes c)\in((B\otimes_K K_m)\otimes_K K_q)\tilde{}$ is in the image of θ. The rest follows easily. By Proposition 4.11, we can write c as $c=\sum_{\gamma\le q}e(c_\gamma)u^\gamma$, so we get that

$$(1\otimes 1\otimes c)=\Big(1\otimes 1\otimes\sum_{\gamma\le q}e(c_\gamma)u^\gamma\Big)=\sum_{\gamma\le q}\big(e(c_\gamma)\otimes 1\otimes u^\gamma\big).$$

Thus

$$\theta\Big(\sum_{\gamma\le q}(e(c_\gamma)\otimes u^\gamma\otimes 1)\Big)=(1\otimes 1\otimes c).\qquad\blacksquare$$

Remark 4.25: Let X be a K-scheme. As in Remark 2.8, for $0\le m\le n\le\infty$, the canonical maps $f_{mn}:\mathrm{HS}^m_{A/(K,\underline{D})}\to\mathrm{HS}^n_{A/(K,\underline{D})}$ determine a directed system of morphisms

$$f_{mn} : \mathrm{HS}^m_{X/(K,\underline{D})} \longrightarrow \mathrm{HS}^n_{X/(K,\underline{D})}.$$

In terms of schemes, the f_{mn} give morphisms

$$\pi_{nm} : P_n(X/(K,\underline{D})) \longrightarrow P_m(X/(K,\underline{D}))$$

which also form a directed system. Exactly as above, we also have

$$\mathrm{HS}^\infty_{X/(K,\underline{D})} = \varinjlim_{i\in\mathbb{N}} \mathrm{HS}^i_{X/(K,\underline{D})}$$

and

$$P_\infty(X/(K,\underline{D})) = \varprojlim_{i\in\mathbb{N}} P_i(X/(K,\underline{D})).$$

Functorial properties. There are many functorial properties of these constructions, precisely as discussed on page 257.

Lemma 4.26: *Let A be a $(K,\underline{D})$-algebra, $(K',\underline{D})$ a $\mathcal{D}$-extension field of K, and $A' = A \otimes_K K'$. Then $\mathrm{HS}^m_{A'/(K',\underline{D})} \cong \mathrm{HS}^m_{A/(K,\underline{D})} \otimes_K K'$ as A'-algebras.*

Proof. Let ϕ be the map from $\mathrm{HS}^m_{A'/(K',\underline{D})}$ to $\mathrm{HS}^m_{A/(K,\underline{D})} \otimes_K K'$ that sends $d_\alpha(a \otimes c), \alpha \le m, a \in A, c \in K$, to $\sum_{\beta+\gamma=k}(d_\beta a \otimes 1)(1 \otimes D_\gamma c))$. It is clear that ϕ is an isomorphism. ∎

Corollary 4.27: *Let $(K,\underline{D})$ be a $\mathcal{D}$-field, and let $(K',\underline{D})$ be a $\mathcal{D}$-field extension. Then for all K-schemes X and all m,*

$$P_m(X \times_K \operatorname{Spec} K') \cong P_m(X) \times_K \operatorname{Spec} K'.$$

As above, if $f : X \to X'$ is a morphism of K-schemes, then there is an induced map $P_m(f) : P_m(X) \to P_m(X')$ between their prolongations.

Lemma 4.28: *Let X, X' be K-schemes, and $f : X \to X'$ a closed immersion. Then $P_m(f) : P_m(X) \to P_m(X')$ is also a closed immersion.*

Proposition 4.29: *Let $f : X \to Y$ be an étale morphism of schemes over a $\mathcal{D}$-field $(K,\underline{D})$. Then for all m,*

$$P_m(X) \cong X \times_Y P_m(Y).$$

Remark 4.30: Notice by Remark 4.18 that for any q and any $m < \infty$, $\dim(P_m(\mathbb{A}^q)) = q \cdot c_{n,m}$.

Proposition 4.31: *Let X be a smooth scheme over the $\mathcal{D}$-field $(K, \underline{D})$ of dimension q. Then for all m, $P_m(X)$ is an $\mathbb{A}^{q\cdot c_{n,m}}$-bundle over X. (That is, X can be covered by open sets U such that $P_m(U) \cong U \times_K \mathbb{A}^{q\cdot c_{n,m}}$.*

Proof. By hypothesis, $X \to \operatorname{Spec} K$ is a smooth map, so, by [EGA], this implies that there is a covering of X by open sets U_i, such that for all i, the following diagram commutes

$$\begin{array}{ccc} U_i & \xrightarrow{g_i} & \mathbb{A}^n \\ \downarrow & & \downarrow \\ K & \xrightarrow{=} & K \end{array}$$

and g_i is étale. By the previous proposition, $P_m(U_i) \cong U_i \times \mathbb{A}^{q\cdot c_{n,m}}$, as desired. ∎

Corollary 4.32: *Let X be a smooth scheme over the $\mathcal{D}$-field $(K, \underline{D})$ of dimension q. Then for all m, $P_{m+1}(X)$ is an $\mathbb{A}^{q(c_{n,m}-c_{n,m-1})}$-bundle over $P_m(X)$.*

4.2. $\mathcal{D}$-Schemes. We generalize material from Section 3, which is straightforward.

Definition 4.33: Let $(K, \underline{D})$ be a $\mathcal{D}$-field. A **$\mathcal{D}$-scheme** over $(K, \underline{D})$ is a K-scheme X such that $\mathcal{O}_X$ is a structure sheaf of $\mathcal{D}$-$(K, \underline{D})$-algebras. A **morphism** of $\mathcal{D}$-schemes is a morphism of R-schemes such that the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves of $(R, \underline{D})$-algebras.

Proposition 4.34: *Let $(A, \underline{D})$ be a $\mathcal{D}$-$(K, \underline{D})$-algebra. There exists a $\mathcal{D}$-scheme $X = \mathcal{D}\text{-Spec}(A, \underline{D})$ such that, forgetting the $\mathcal{D}$-structure on X, X is isomorphic to* $\operatorname{Spec} A$.

Proposition 4.35: *Let $(A, \underline{D})$ be a $\mathcal{D}$-ring, and $(X, \mathcal{O}_X)$ a* $\mathcal{D}$-scheme. *Then there is a bijection:*

$$\chi : \operatorname{Hom}_{\mathcal{D}-\mathrm{Sch}}(X, \operatorname{Spec} A) \longrightarrow \operatorname{Hom}_{\mathcal{D}-\mathrm{Ring}}(A, \Gamma(X, \mathcal{O}_X)).$$

Remark 4.36: Let $X \subseteq \mathbb{A}^q$ be an affine K-scheme,

$$\Gamma(X, \mathcal{O}_X) = K[x_i]_{i=1,\ldots,q}/(f_j)_{j\in J}.$$

For all $m \leq \infty$, $P_m(X)$ is the closed subscheme of

$$\mathbb{A}^{q\cdot c_{n,m}} = \operatorname{Spec}(K[D_\alpha x_i]_{i=1,\ldots,q,\alpha\leq m})$$

with

$$\Gamma(P_m(X), \mathcal{O}_{P_m(X)}) = K[D_\alpha x_i]_{i=1,\dots,q,\alpha\leq m}/(D_\alpha f_j)_{j\in J,\alpha\leq m}.$$

(This follows from Proposition 4.19.) In particular, for every closed point $(a_1, \dots, a_q) \in X$, the point $(D_\alpha a_i)_{i=1,\dots,q,\alpha\leq m}$ is in $P_m(X)$. The canonical projection from $P_m(X)$ to X maps a closed point $(a_{i,\alpha})_{i=1,\dots,q,\alpha\leq m}$ to its first q coordinates, $(a_{i,0})_{i=1,\dots,q}$.

Proposition 4.37: *Let $(K, \underline{D})$ be a $\mathcal{D}$-field. The prolongation functor, that takes a K-scheme X to the $\mathcal{D}$-scheme $P_\infty(X)$, is the right adjoint to the forgetful functor $Y \mapsto Y^!$ from $\mathcal{D}$-schemes to K-schemes.*

As before, if X is a $\mathcal{D}$-scheme, we define a K-rational point of X to be a $\mathcal{D}$-scheme homomorphism from $\mathcal{D}$-Spec K to X. Of course, a $\mathcal{D}$-morphism $f : X \to Y$ naturally induces a map between their K-rational points. The previous proposition immediately implies that there is a natural bijection between K-rational points of X and of $P_\infty(X)$.

Definition 4.38: Let X, Y be K-schemes, and $f : P_\infty(X) \to P_\infty(Y)$ be a $\mathcal{D}$-morphism. The natural bijections

$$\chi : \mathrm{Hom}_K(\mathrm{Spec}\, K, X) \longrightarrow \mathrm{Hom}_{(K,\underline{D})}(\mathcal{D}\text{-Spec}\, K, P_\infty(X))$$

and

$$\zeta : \mathrm{Hom}_K(\mathrm{Spec}\, K, Y) \longrightarrow \mathrm{Hom}_{(K,\underline{D})}(\mathcal{D}\text{-Spec}\, K, P_\infty(Y))$$

and the induced map

$$\hat{f} : \mathrm{Hom}_{(K,\underline{D})}(\mathcal{D}\text{-Spec}\, K, P_\infty(X)) \longrightarrow \mathrm{Hom}_{(K,\underline{D})}(\mathcal{D}\text{-Spec}\, K, P_\infty(Y))$$

determine a (set theoretic) map from K-rational points of X to those of Y, given by $\zeta^{-1} \circ \hat{f} \circ \chi$.

A **$\mathcal{D}$-polynomial map** from X to Y is a map on K-rational points of the form $\zeta^{-1} \circ \hat{f} \circ \chi$, for some $\mathcal{D}$-morphism $f : P_\infty(X) \to P_\infty(Y)$.

Schemes X and Y are **$\mathcal{D}$-polynomially isomorphic** if there are $\mathcal{D}$-polynomial maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$.

Remark 4.39: Let $X = \mathrm{Spec}(K[x_i]_{i\leq q}/(f_j)_{j\in J})$, so that

$$P_\infty(X) = \mathrm{Spec}(K[d_\alpha x_i]_{i\leq q,\alpha<\infty}/(f_j)_{j\in J}).$$

The bijection χ takes $h \in \mathrm{Hom}_K(\mathrm{Spec}\, K, X)$, which is determined by $(b_i)_{i\le q} = h(x_i)_{i\le q}$ to $H \in \mathrm{Hom}_{(K,\underline{D})}(\mathrm{Spec}\, K, P_\infty(X))$ determined by $(D_\alpha b_i)_{i\le q,\alpha<\infty} = H(d_\alpha x_i)_{i\le q,\alpha<\infty}$.

Proposition 4.40: *Let X be a K-scheme, and $m < \infty$. There exists a $\mathcal{D}$-polynomial map $\nabla_m : X \to P_m(X)$ that is a section of the canonical projection $p_m : P_m(X) \to X$.*

Let $f : X \to Y$ be a morphism of K-schemes. Considering f and $P_m(f)$ as maps on K-rational points, the following diagram commutes.

$$\begin{array}{ccc} P_m(X) & \xrightarrow{P_m(f)} & P_m(Y) \\ {\scriptstyle \nabla_m}\uparrow & & \uparrow{\scriptstyle \nabla_m} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

Proof. (Again, the proof is similar to the corresponding result for fields with a single derivation.) By the adjointness of $P_\infty(-)$ and $(-)^!$, there is a natural bijection

$$\mathrm{Hom}_K((P_\infty(X))^!, P_m(X)) \simeq \mathrm{Hom}_{(K,\underline{D})}(P_\infty(X), P_\infty(P_m(X))).$$

Let $f : P_\infty(X) \to P_\infty(P_m(X))$ be the $\mathcal{D}$-morphism corresponding to the canonical projection from $(P_\infty(X))^!$ to $P_m(X)$, and let ∇_m be the $\mathcal{D}$-polynomial map corresponding to f. We show that ∇_m has the desired properties.

It suffices to check locally, so suppose that X is given as $\mathrm{Spec}(K[x_i]_{i\le q}/(f_j)_{j\in J})$. By Remark 4.36,

$$\begin{aligned} P_m(X) &= \mathrm{Spec}(K[d_\alpha x_i]_{\alpha\le m, i\le q}/(d_\alpha f_j)_{\alpha\le m, j\in J}) \\ P_\infty(X) &= \mathrm{Spec}(K[d_\alpha x_i]_{\alpha<\infty, i\le q}/(d_\alpha f_j)_{\alpha<\infty, j\in J}) \\ P_\infty(P_m(X)) &= \mathrm{Spec}(K[d_\beta d_\alpha x_i]_{\alpha\le m,\beta<\infty, i\le q}/(g_h)_{h\in H}) \end{aligned}$$

where $(g_h)_{h\in H}$ is the ideal generated by $(d_\beta d_\alpha f_j)_{j\in J,\alpha\le m, l<\infty}$. The $\mathcal{D}$-morphism from $P_\infty(X)$ to $P_\infty(P_m(X))$, corresponding to the projection morphism from $P_\infty(X)$ to $P_m(X)$ is determined by the $\mathcal{D}$-algebra homomorphism

$$K[d_\beta d_\alpha x_i]_{\alpha\le m,\beta<\infty, i\le q}/(g_h)_{h\in H} \longrightarrow K[d_\alpha x_i]_{\alpha<\infty, i\le q}/(d_\alpha f_j)_{\alpha<\infty, j\in J}$$

such that

$$d_\beta d_\alpha x_i \mapsto \binom{\alpha_1+\beta_1}{\alpha_1}\cdots\binom{\alpha_n+\beta_n}{\alpha_n} d_{\alpha+\beta} x_i.$$

One can then see that this determines the $\mathcal{D}$-polynomial map from X to $P_m(X)$ that takes the closed point $(a_i)_{i\le n}$ to $(D_\alpha a_i)_{\alpha\le m, i\le n}$. By Remark 4.36, this is a section of π_m.

Next we argue that $P_m(f)\circ\nabla_X = \nabla_Y\circ f$. It suffices to prove this for affine schemes, so assume that $X = \operatorname{Spec} K[\overline{x}]/I$ and $Y = \operatorname{Spec} K[\overline{y}]/J$. Let $S = K[\overline{x}]/I$ and $R = K[\overline{y}]/J$, and let f also denote the homomorphism from R to S corresponding to $f : X \to Y$. A K-rational point of X corresponds to a homomorphism g from S to K, which is determined by the image of $\overline{x}$, so we think of a K-rational point as a tuple $\overline{a} = g(\overline{x})$ of elements of K. Also, $P_m(X) = \operatorname{Spec}\mathrm{HS}^m_{S/(K,\underline{D})}$ is affine, and $\mathrm{HS}^m_{S/(K,\underline{D})}$ is generated by $(d_\alpha x)_{x\in\overline{x},\alpha\le m}$. We have seen that $\nabla_X(\overline{a}) = (\overline{a}, D_1(\overline{a}),\dots,D_m(\overline{a}))$. To be more precise, $\nabla_X(\overline{a})$ is the K-rational point of $P_m(X)$ that corresponds to the map that sends $d_\alpha x \in \mathrm{HS}^m_{S/(K,\underline{D})}$ to $D_\alpha(g(x))\in K$, for each $x\in\overline{x},\alpha\le m$.

Let $f(\overline{a}) = \overline{b}\in Y$, $\overline{b} = (g\circ f(y))_{y\in\overline{y}}$. Again, $\nabla_Y(\overline{b}) = (\overline{b}, D_1(\overline{b}),\dots,D_m(\overline{b}))$. As a map of K-algebras, $P_m(f)$ is the map that sends $d_\alpha y$ to $d_\alpha f(y)$, for $y\in\overline{y},\alpha\le m$. Thus,

$$P_m(f)(\overline{a}, D_1(\overline{a}),\dots,D_m(\overline{a})) = (\overline{b}, D_1(\overline{b}),\dots,D_m(\overline{b}))$$

as desired. ∎

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